Tutorial 2 Solutions
September 18/19, 2008

1. (a) The sample space consists of all ways of drawing 7 elements out of a 52-element set, so it contains $\binom{52}{7}$ possible outcomes. Let us count those outcomes that involve exactly 3 aces. We are free to select any 3 out of the 4 aces, and any 4 out of the 48 remaining cards, for a total of $\binom{4}{3}\binom{48}{4}$ choices. Thus,

$$\mathbf{P}(\text{7 cards include exactly 3 aces}) = \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}}.$$

(b) Proceeding similar to part (a), we obtain

$$\mathbf{P}(\text{7 cards include exactly 2 kings}) = \frac{\binom{4}{2}\binom{48}{5}}{\binom{52}{7}}.$$

(c) If A and B stand for the events in parts (a) and (b), respectively, we are looking for $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$. The event $A \cap B$ (having exactly 3 aces and exactly 2 kings) can occur by choosing 3 out of the 4 available aces, 2 out of the 4 available kings, and 2 more cards out of the remaining 44. Thus, this event consists of $\binom{4}{3}\binom{4}{2}\binom{44}{2}$ distinct outcomes. Hence,

$$\mathbf{P}(\text{7 cards include 3 aces and/or 2 kings}) = \frac{\binom{4}{3}\binom{48}{4} + \binom{4}{2}\binom{48}{5} - \binom{4}{3}\binom{4}{2}\binom{44}{2}}{\binom{52}{7}}.$$

2. (a) The scalar a must satisfy

$$1 = \sum_x p_X(x) = \frac{1}{a}\sum_{x=-3}^{3} x^2,$$

so

$$a = \sum_{x=-3}^{3} x^2 = (-3)^2 + (-2)^2 + (-1)^2 + 1^2 + 2^2 + 3^2 = 28.$$

We also have $\mathbf{E}[X] = 0$ because the PMF is symmetric around 0.

(b) If $z \in \{1, 4, 9\}$, then

$$p_Z(z) = p_X(\sqrt{z}) + p_X(-\sqrt{z}) = \frac{z}{28} + \frac{z}{28} = \frac{z}{14}.$$

Otherwise $p_Z(z) = 0$.

(c) $\text{var}(X) = \mathbf{E}[Z] = \sum_z z p_Z(z) = \sum_{z \in \{1,4,9\}} \frac{z^2}{14} = 7.$

(d) We have

$$\begin{aligned} \text{var}(X) &= \sum_x (x - \mathbf{E}[X])^2 p_X(x) \\ &= 1^2 \cdot (p_X(-1) + p_X(1)) + 2^2 \cdot (p_X(-2) + p_X(2)) + 3^2 \cdot (p_X(-3) + p_X(3)) \\ &= 2 \cdot \frac{1}{28} + 8 \cdot \frac{4}{28} + 18 \cdot \frac{9}{28} \\ &= 7. \end{aligned}$$

3. **Lottery.** We assume all choices for length-r lottery numbers out of n integers are equally-likely and, therefore, appeal to the discrete uniform probability law and counting arguments. In other words, given A denotes the event of interest, we employ

$$\mathbf{P}(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega} \equiv \frac{\text{number of favorable outcomes}}{\text{number of total outcomes}}$$

in each part of this problem. We also assume that winning this particular lottery does not depend on the order that the integers are chosen and that $n >> r$.

(a) Event A corresponds to the subset L being drawn in increasing order. Note that any particular length-r subset L may result from drawing the same r integers in $r!$ different orderings, and only one of these orderings can be in increasing order. Combining this observation with the total probability theorem,

$$\mathbf{P}(A) = \sum_{\text{subsets}} \mathbf{P}(\text{subset}) \underbrace{\mathbf{P}(A|\text{subset})}_{\frac{1}{r!}} = \frac{1}{r!} \underbrace{\sum_{\text{subsets}} \mathbf{P}(\text{subset})}_{1} = \boxed{\frac{1}{r!}} \quad .$$

(b) Event A corresponds to when the r numbers in L are the same as those chosen by you. Clearly, among all the $\binom{n}{r}$ choices only one is favorable and hence the probability is

$$\mathbf{P}(A) = \boxed{\frac{1}{\binom{n}{r}}} \quad .$$

(c) Event A corresponds to having exactly k of the numbers in L match the numbers chosen by you. The number of favorable outcomes is then the *product* of (i) the number of unique length-k subsets of L, or $\binom{r}{k}$, and (ii) the number of ways we could choose the remaining $r-k$ integers from the set that the lottery did *not* select, or $\binom{n-r}{r-k}$. Therefore the desired probability is equal to

$$\mathbf{P}(A) = \boxed{\frac{\binom{r}{k}\binom{n-r}{r-k}}{\binom{n}{r}}} \quad .$$

(d) Event A corresponds to the length-r subset L containing no consecutive integers. Envision the n integers ordered in a row and suppose, each time we pick an integer, that we somehow mark the integer; say, for example, that we circle that integer. This implies that any unique subset L is identified with a unique placement of r circles in n spaces. It follows that the number of total outcomes pertaining to the selection of L is $\binom{n}{r}$ and it remains to determine the number of favorable outcomes, or the ones with no consecutive integers.

The number of favorable outcomes is equivalent to the number of ways of placing the r circles in the n spaces such that no circle is adjacent to any other circle. In any such placement of the r circles, there must be exactly $r-1$ spaces that act as "dividers" between the placed circles; in other words, there must be $r-1$ integers that cannot be chosen to result in r chosen nonconsecutive integers.

We can construct a bijection from the choices of numbers that are nonconsecutive to those numbers which are picked in the following manner. Except for the largest number chosen, you circle the number you want AND the following one. For each of these $r-1$ circled pairs, think of them as just one lump. It is easy to show that this is a bijection. Thus, in effect, the placement of these r circles is restricted to be out of only $n-(r-1)$ spaces, which implies the number of favorable outcomes is $\binom{n-r+1}{r}$, yielding

$$\mathbf{P}(A) = \boxed{\frac{\binom{n-r+1}{r}}{\binom{n}{r}}}$$

(e) Event A corresponds to the length-r subset L containing exactly one pair of consecutive integers. The total number of possible outcomes here can be visualized using the same procedure as discussed in part (d) but it remains to determine the number of favorable outcomes.

The number of such subsets can be computed as before, except we recognize that at least one choice is automatically tied to its neighbor. In other words, we are free to choose only $r-1$ integers from the $n-r+1$ allowable choices. But we also must account for the fact that the forced choice can be associated with any one of the $r-1$ free choices. Hence, the number of favorable outcomes is

$$(r-1)\binom{n-r+1}{r-1} \quad ,$$

yielding the desired probability

$$\mathbf{P}(A) = \boxed{(r-1)\frac{\binom{n-r+1}{r-1}}{\binom{n}{r}}} \quad .$$