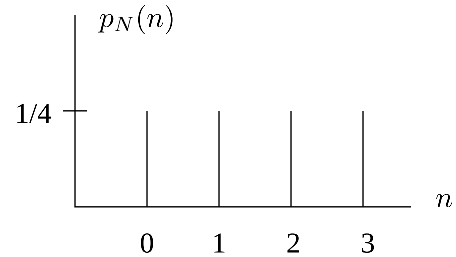
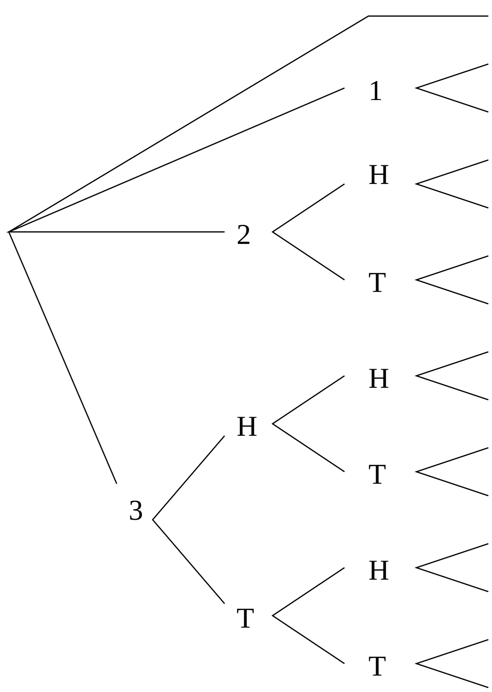
Tutorial 3: Solutions [1]
September 25/26, 2008

1. (a) The first part can be completed without reference to anything other than the die roll:



(b) A sequential sample space for this experiment is as follows:



	Pr	k	n
0	1/4	0	0
H	1/8	1	1
T	1/8	0	1
H	1/16	2	2
T	1/16	1	2
H	1/16	1	2
T	1/16	0	2
H	1/32	3	3
T	1/32	2	3
H	1/32	2	3
T	1/32	1	3
H	1/32	2	3
T	1/32	1	3
H	1/32	1	3
T	1/32	0	3

The sample space conditional on $N = 2$ is:

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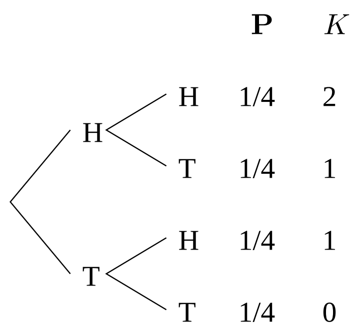
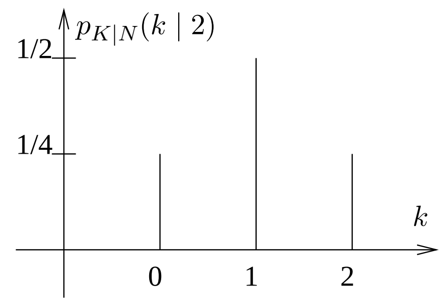
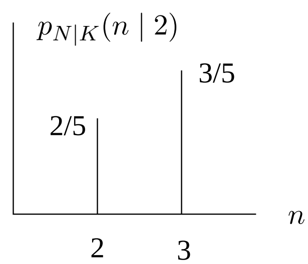




(c) The remaining parts can be done by straightforward but tedious collection of the relevant events in the original sample space. For example,

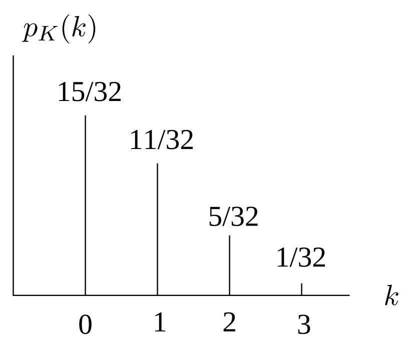
$$p_{N|K}(2 \mid 2) = \frac{\mathbf{P}(N = 2,\ K = 2)}{\mathbf{P}(K = 2)} = \frac{1/16}{1/16 + 1/32 + 1/32 + 1/32} = 2/5.$$

The remaining terms of the PMF are computed in the same way.



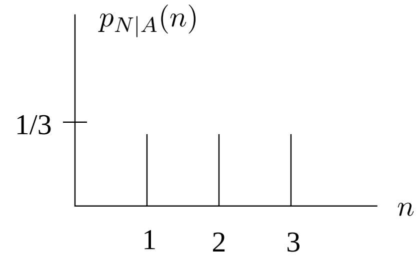
(d) For example,

$$p_K(0) = \mathbf{P}(K = 0) = 1/4 + 1/8 + 1/16 + 1/32 = 15/32.$$



(e) Let A denote the event that K is an odd number. The first term of the PMF for N conditional on A is computed as

$$p_{N|A}(0) = \frac{\mathbf{P}(\{N = 0\} \cap A)}{\mathbf{P}(A)} = \frac{\mathbf{P}(N = 0,\ K \text{ odd})}{\mathbf{P}(K \text{ odd})} = 0.$$



(f) $\mathbf{E}[K|N=2]=1$, $\text{var}[K|N=2]=0.5$

2. Consider a sequence of six independent rolls of this die, and let x_i be the random variable corresponding to the ith roll.

   (a) What is the probability that exactly three of the rolls have an outcome equal to three? Each roll x_i can either be a three with probability 1/4 or not a three with probability 3/4. There are $\binom{6}{3}$ ways of placing the threes in the sequence of 6 rolls. After we require that a three go in each of these spots, which takes probablity $(\frac{1}{4})^3$ our only remaining condition is that either a one or a two go in the other three spots, which takes probability $(\frac{3}{4})^3$. So the probability of exactly three rolls in a sequence of 6 independent rolls is $\boxed{\binom{6}{3}(\frac{1}{4})^3(\frac{3}{4})^3}$.

   (b) What is the probability that the first roll is a 1, given that exactly two of the six rolls had an outcome of 1? The probability of obtaining a one on a single roll is 1/2, and the probability of obtaining a 2 or 3 on a single roll is also 1/2. For the purposes of solving this problem we treat obtaining a 2 or 3 as an equivalent outcome. We know that there are $\binom{6}{2}$ ways of rolling exactly 2 ones. Of these $\binom{6}{2}$ ways exactly $\binom{5}{1}=5$ ways result in a one in the first roll, since we can place the remaining one in any of the 5 remaining rolls. The rest of the rolls must be either two or three. Thus the probability that the first roll is a one given exactly 2 rolls had an outcome of one is $\boxed{\frac{5}{\binom{6}{2}}}$.

   (c) We are now told that exactly three of the rolls resulted in one and exactly three resulted in 2. What is the probability of the outcome 121212? We want to find

$$\mathbf{P}(121212\,|\,\text{exactly 3 ones and 3 twos}) = \frac{\mathbf{P}(121212)}{\mathbf{P}(\text{exactly 3 ones and 3 twos})}.$$

   Any particular sequence of three ones and three twos will have the same probability: $(\frac{1}{2})^3(\frac{1}{4})^3$. There are $\binom{6}{3}$ possible rolls with exactly three ones and three twos. Therefore $\mathbf{P}(121212\,|\,\text{exactly 3 ones and 3 twos}) = \boxed{\frac{1}{\binom{6}{3}}}$.

   (d) Conditioned on the event that at least one roll resulted in 3, find the conditional PMF of the number of 3's. Let A be the event that at least one roll results in a three. Then $\mathbf{P}(A) = 1 - \mathbf{P}(\text{no rolls resulted in three}) = 1 - (\frac{3}{4})^6$. Now let k be the random variable representing the number of threes in the 6 rolls. Our unconditional PMF $p_K(k)$ for k is given by

$$p_K(k) = \binom{6}{k}\frac{1}{4}^k\frac{3}{4}^{6-k}.$$

We find the conditional PMF $p_{k|A}(k|A)$ for k using the definition of conditional probability:

$$p_{K|A}(k|A) = \frac{\mathbf{P}(K = k, A)}{\mathbf{P}(A)}.$$

Thus we obtain

$$p_{K|A}(k|A) = \begin{cases} \frac{1}{1-(3/4)^6}\binom{6}{k}(\frac{1}{4})^k(\frac{3}{4})^{6-k} & \text{for } k = 1, 2, \ldots, 6, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $p_{K|A}(0|A) = 0$ because the event $k = 0$ and the event A are mutually exclusive. Thus the probability of their intersection, which appears in the numerator in the definition of the conditional PMF, is zero.

3. There is not enough information to answer this question. We need to know the a priori probability that a candidate is qualified. Given this information, let Q be the event that someone is qualified, Q^c the event that someone is unqualified, and A the event that the 20 questions correctly determine whether the candidate is qualified or not. Then:

$$\begin{aligned} P(A) &= P(A \cap Q) + P(A \cap Q^c) \\ &= P(A|Q) \cdot P(Q) + P(A|Q^c) \cdot P(Q^c) \\ &= P(Q) \cdot \sum_{i=15}^{20} \binom{20}{i} \cdot p^i(1-p)^{20-i} + P(Q^c) \cdot \sum_{i=6}^{20} \binom{20}{i} \cdot p^i(1-p)^{20-i} \end{aligned}$$