Tutorial 4 Solutions
October 2/3, 2008

1. Since we are trying repeatedly to obtain a value greater than $\frac{7}{2}$, this is a geometric distribution. Its parameter is given by $\mathbf{P}(X > \frac{7}{2})$. Therefore, the expected number of trials is $\frac{1}{\mathbf{P}(X>\frac{7}{2})}$.

$$\begin{aligned}\mathbf{P}(X > \frac{7}{2}) &= \int_{\frac{7}{2}}^{+\infty} f_X(x)\,dx \\ &= \int_{\frac{7}{2}}^{4} \frac{1}{8}\,dx \\ &= \frac{1}{16}\end{aligned}$$

So, the expected number of trials is 16.

2. (a) We find the expectation by integrating:

$$\begin{aligned}\mathbf{E}[Z] &= \int_{-\infty}^{+\infty} z f_Z(z)\,dz \\ &= \int_0^1 5z(1-z)^4\,dz\end{aligned}$$

Making the change of variables $u = 1 - z$,

$$\begin{aligned} &= -\int_1^0 \left(5u^4 - 5u^5\right)\,dz \\ &= \left[u^5 - \frac{5}{6}u^6\right]_0^1 \\ &= \frac{1}{6} \\ &\approx \$16,666\end{aligned}$$

We find the variance first by computing the second moment:

$$\begin{aligned}\mathbf{E}[Z^2] &= \int_{-\infty}^{+\infty} z^2 f_Z(z)\,dz \\ &= \int_0^1 5z^2(1-z)^4\,dz\end{aligned}$$

Making the change of variables $u = 1 - z$,

$$\begin{aligned} &= -\int_1^0 \left(5u^6 - 10u^5 + 5u^4\right)\,dz \\ &= \left[\frac{5}{7}u^7 - \frac{5}{3}u^6 + u^5\right]_0^1 \\ &= \frac{1}{21}\end{aligned}$$

Therefore, the variance is

$$\begin{aligned}\mathbf{E}[Z^2] - (\mathbf{E}[Z])^2 &= \frac{1}{21} - \frac{1}{36} \\ &= \frac{5}{252} \\ &\approx \$1984\end{aligned}$$

(b) The question asks us to find a real number x such that:

$$\mathbf{P}(Z > x) \leq 0.01$$

Now, for $0 \leq x \leq 1$

$$\begin{aligned}\mathbf{P}(Z > x) &= 1 - \mathbf{P}(Z \leq x) \\ &= 1 - \int_0^x f_Z(z)dz \\ &= 1 - \int_0^x 5((1-z)^4)dz \\ &= 1 + (1-x)^5 - 1\end{aligned}$$

and from here we easily compute that approximately, we need $x \geq 0.6019$, or \$60190.

3. (a) The CDF of a normal random variable X with mean μ and variance σ can be obtained from the standard normal table by standardizing X and redefining the probability in terms of Y, the standard normal random variable.
The standard normal random variable:

$$Y = \frac{X - \mu}{\sigma}$$

Therefore,

$$\begin{aligned}P(X \leq \mu + \sigma) &= P(X - \mu \leq \sigma) \\ &= P(\frac{X-\mu}{\sigma} \leq 1) \\ &= P(Y \leq 1) \\ &= \Phi(1) \\ &= 0.8413\end{aligned}$$

Similarly,

$$\begin{aligned}P(X \leq \mu - \sigma) &= P(X - \mu \leq -\sigma) \\ &= P(\frac{X-\mu}{\sigma} \leq -1) \\ &= P(Y \leq -1) \\ &= \Phi(-1) \\ &= 1 - \Phi(1) \\ &= 1 - 0.8413 \\ &= 0.1587\end{aligned}$$

And,

$$\begin{aligned} P(X \leq \mu + 2\sigma) &= P(X - \mu \leq +2\sigma) \\ &= P(\frac{X-\mu}{\sigma} \leq 2) \\ &= P(Y \leq 2) \\ &= \Phi(2) \\ &= 0.9772 \end{aligned}$$

(b) Using the same approach as part (a).

$$\begin{aligned} P(\mu - \sigma \leq X \leq \mu + \sigma) &= P(-\sigma \leq X - \mu \leq \sigma) \\ &= P(-1 \leq \frac{X-\mu}{\sigma} \leq 1) \\ &= P(-1 \leq Y \leq 1) \\ &= \Phi(1) - \Phi(-1) \\ &= \Phi(1) - (1 - \Phi(1)) \\ &= 2\Phi(1) - 1 \\ &= 0.6826 \end{aligned}$$

Similarly,

$$\begin{aligned} P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) &= P(-2\sigma \leq X - \mu \leq 2\sigma) \\ &= P(-2 \leq \frac{X-\mu}{\sigma} \leq 2) \\ &= P(-2 \leq Y \leq 2) \\ &= \Phi(2) - \Phi(-2) \\ &= \Phi(2) - (1 - \Phi(2)) \\ &= 2\Phi(2) - 1 \\ &= 0.9544 \end{aligned}$$