

Tutorial 5
October 9/10, 2008

1. For each wheel, we can find the probability of obtaining a value smaller than $1/4$.

$$\begin{aligned}\mathbf{P}\left(X \leq \frac{1}{4}\right) &= \int_0^{\frac{1}{4}} f_X(x) dx = \frac{1}{4} \\ \mathbf{P}\left(Y \leq \frac{1}{4}\right) &= \int_0^{\frac{1}{4}} f_Y(y) dy = \frac{3}{4}\end{aligned}$$

(a) Using Bayes' Rule gives

$$\begin{aligned}\mathbf{P}\left(A \mid Z < \frac{1}{4}\right) &= \frac{\mathbf{P}(A)\mathbf{P}(Z \leq \frac{1}{4} \mid A)}{\mathbf{P}(Z \leq \frac{1}{4})} = \frac{\mathbf{P}(A)\mathbf{P}(X \leq \frac{1}{4})}{\mathbf{P}(Z \leq \frac{1}{4})} \\ &= \frac{\mathbf{P}(A)\mathbf{P}(X \leq \frac{1}{4})}{\mathbf{P}(A)\mathbf{P}(Z \leq \frac{1}{4} \mid A) + \mathbf{P}(B)\mathbf{P}(Z \leq \frac{1}{4} \mid B)} \\ &= \frac{\mathbf{P}(A)\mathbf{P}(X \leq \frac{1}{4})}{\mathbf{P}(A)\mathbf{P}(X \leq \frac{1}{4}) + \mathbf{P}(B)\mathbf{P}(Y \leq \frac{1}{4})} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{4}}{\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{4}} = \frac{1}{4}.\end{aligned}$$

(b) Here we use the continuous Bayes' rule:

$$\begin{aligned}\mathbf{P}\left(A \mid Z = \frac{1}{4}\right) &= \frac{f_{Z|A}\left(\frac{1}{4}\right) \mathbf{P}(A)}{f_Z\left(\frac{1}{4}\right)} \\ &= \frac{f_X\left(\frac{1}{4}\right) \frac{1}{2}}{\mathbf{P}(A)f_{Z|A}\left(\frac{1}{4}\right) + \mathbf{P}(B)f_{Z|B}\left(\frac{1}{4}\right)} \\ &= \frac{f_X\left(\frac{1}{4}\right) \frac{1}{2}}{\frac{1}{2}f_X\left(\frac{1}{4}\right) + \frac{1}{2}f_Y\left(\frac{1}{4}\right)} \\ &= \frac{1}{4}\end{aligned}$$

2. (a) Because X and Y are independent, we know $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. Thus, the joint pdf is

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-2y} & 0 \leq x \leq 1, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbf{P}(Y \geq X)$ is simply $1 - \mathbf{P}(Y \leq X)$. The region where $Y \leq X$ is a triangle, where x is from 0 to 1 and y is from 0 to x .

$$\begin{aligned}
 \mathbf{P}(Y \geq X) &= \iint_{y \geq x} f_{X,Y}(x,y) dx dy \\
 &= 1 - \int_0^1 \int_0^x 2e^{-2y} dy dx \\
 &= 1 - \int_0^1 1 - e^{-2x} dx \\
 &= \frac{1}{2} - \frac{e^{-2}}{2}
 \end{aligned}$$

(b) Conditioning on $Y = y$, find the PDF of Z

$$\begin{aligned}
 f_{Z|Y=y}(z) &= f_{X+Y|Y=y}(x+y) \\
 &= \begin{cases} 1 & y \leq z \leq 1+y \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

3. In order to solve this problem, it is useful to begin by deriving the marginal distributions $f_X(x)$ and $f_Y(y)$. The marginal distributions are obtained by integrating the joint distribution along the X and Y axes and is shown in the following figure.

(a) The conditional PDF $f_{Y|X}(y|x)$ is given by

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

A good way to visualize the conditional PDF $f_{Y|X}(y|x)$ is to imagine a vertical slice of the joint PDF at $X = x$. Essentially, the conditional PDF has the same shape as the joint PDF except for a scaling factor $f_X(x)$ which ensures that,

$$\int f_{Y|X}(y|x) dy = 1$$

Similarly the conditional PDF $f_{X|Y}(x|y)$ is obtained using,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

To visualize $f_{X|Y}(x|y)$, imagine a horizontal slice through the joint PDF at $Y = y$. Again, the conditional PDF has the same shape except for the scaling factor of $f_Y(y)$. The conditional PDFs are as shown in the figure below.

(b) X and Y are **NOT** independent since $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$. Also, from the figures we have $f_{X|Y}(x|y) \neq f_X(x)$ and $f_{Y|X}(y|x) \neq f_Y(y)$.

(c)

$$\begin{aligned}
 f_{X,Y|A}(x,y) &= \begin{cases} \frac{f_{X,Y}(x,y)}{\mathbf{P}(A)} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} \\
 &= \begin{cases} \frac{0.1}{\pi 0.1} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

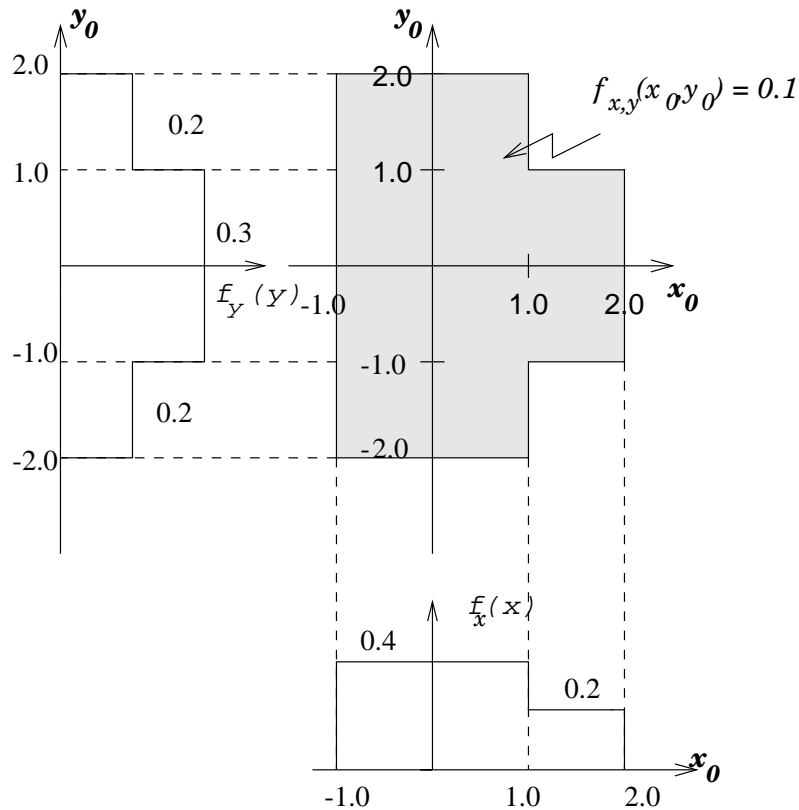


Figure 1: Marginal probabilities $f_X(x)$ and $f_Y(y)$ obtained by integration along the y and x axes respectively

- (d) It can be seen that $f_{X|Y}(x|y)$ is a uniform density for each value of y . Noting the fact that the expected value of a random variable distributed uniformly between a and b is given by $\frac{(a+b)}{2}$, the conditional expected values can be written as

$$\mathbf{E}[X|Y = y] = \begin{cases} 0 & -2.0 \leq y \leq -1.0 \\ \frac{1}{2} & -1.0 \leq y \leq 1.0 \\ 0 & 1.0 \leq y \leq 2.0 \end{cases}$$

Again, noting the fact that the variance of a random variable distributed uniformly between a and b is given by $\frac{(b-a)^2}{12}$, the conditional variance $\sigma_{X|Y}^2$ is given by

$$\text{var}(X|Y = y) = \begin{cases} \frac{4}{12} & -2.0 \leq y \leq -1.0 \\ \frac{9}{12} & -1.0 \leq y \leq 1.0 \\ \frac{4}{12} & 1.0 \leq y \leq 2.0 \end{cases}$$

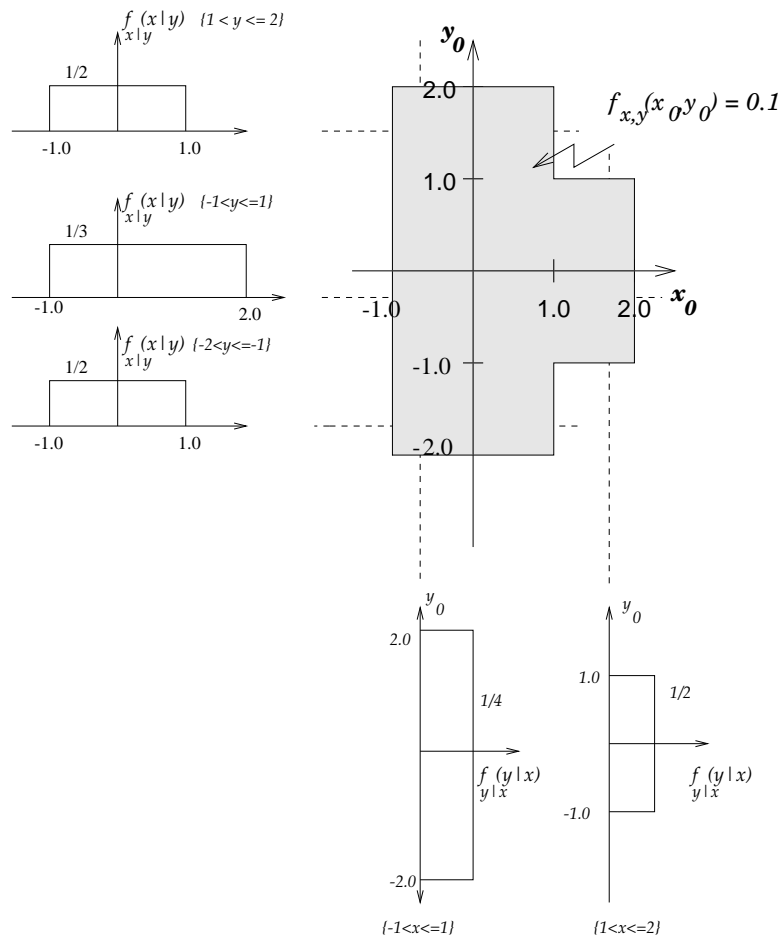


Figure 2: Conditional probabilities $f_X(x)$ and $f_Y(y)$