Tutorial 6: Solutions [1]
October 16/17, 2008

1. (a) Since X, Y and Z are independent, V and W are independent.
Therefore $f_{V,W}(v,w) = f_V(v) * f_W(w)$. Find the seperate CDFs of V and W and then differentiate.

$$\begin{aligned}
F_V(v) &= P(V \le v) = P(XY \le v) = P(XY \le v, Y \le v) + P(XY \le v, Y > v) \\
&= P(Y \le v) + P(X \le v/Y, Y > v) \\
&= \int_0^v f_Y(y)dy + \int_v^1 \int_0^{v/y} f_X(x) f_Y(y) dx dy \\
&= v + \int_v^1 \frac{v}{y} dy = v(1 - \log v), \quad 0 \le v \le 1 \\
F_W(w) &= P(W \le w) = P(Z^2 \le w) = P(Z \le \sqrt{w}) = \sqrt{w}, \quad 0 \le w \le 1 \\
f_V(v) &= \frac{dF_V(v)}{dv} = (1 - \log v) + v(-\frac{1}{v}) = \log(1/v), \quad 0 \le v \le 1 \\
f_W(w) &= \frac{dF_W(w)}{dw} = \frac{1}{2\sqrt{w}}, \quad 0 \le w \le 1 \\
f_{V,W}(v,w) &= \frac{\log(1/v)}{2\sqrt{w}}, \quad 0 \le v, w \le 1
\end{aligned}$$

(b)

$$\begin{aligned}
P(XY \le Z^2) &= P(V \le W) = \int_0^1 \int_0^w \frac{\log(1/v)}{2\sqrt{w}} dv dw = \int_0^1 \frac{v(1 - \log v)}{2\sqrt{w}} |_{v=0}^{w} dw \\
&= \int_0^1 \frac{\sqrt{w}(1 - \log w)}{2} dw = [\frac{w^{3/2}}{3}(\frac{5}{3} - \log w)]_{w=0}^1 = \frac{5}{9}
\end{aligned}$$

2. Let X and Y be the number of flips until Alice and Bob stop, respectively. Thus, $X + Y$ is the total number of flips until both stop. The random variables X and Y are independent geometric random variables with parameters 1/4 and 3/4, respectively. By convolution, we

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have

$$\begin{aligned} p_{X+Y}(j) &= \sum_{k=-\infty}^{\infty} p_X(k) p_Y(j-k) \\ &= \sum_{k=1}^{j-1} (1/4)(3/4)^{k-1}(3/4)(1/4)^{j-k-1} \\ &= \frac{1}{4^j} \sum_{k=1}^{j-1} 3^k \\ &= \frac{1}{4^j} \left(\frac{3^j - 1}{3 - 1} - 1 \right) \\ &= \frac{3}{2} \frac{(3^{j-1} - 1)}{4^j}, \end{aligned}$$

if $j \geq 2$, and 0 otherwise. (Even though $X + Y$ is *not* geometric, it roughly behaves like one with parameter 3/4.)

3. We have

$$\text{cov}(R, S) = \mathbf{E}[RS] - \mathbf{E}[R]\mathbf{E}[S] = \mathbf{E}[WX + WY + X^2 + XY] = \mathbf{E}[X^2] = 1,$$

and

$$\text{var}(R) = \text{var}(S) = 2,$$

so

$$\rho(R, S) = \frac{\text{cov}(R, S)}{\sqrt{\text{var}(R)\text{var}(S)}} = \frac{1}{2}.$$

We also have

$$\text{cov}(R, T) = \mathbf{E}[RT] - \mathbf{E}[R]\mathbf{E}[T] = \mathbf{E}[WY + WZ + XY + XZ] = 0,$$

so that

$$\rho(R, T) = 0.$$

4. Because the covariance remains unchanged when we add a constant to a random variable, we can assume without loss of generality that X and Y have zero mean. We then have

$$\text{cov}(X - Y, X + Y) = \mathbf{E}\big[(X - Y)(X + Y)\big] = \mathbf{E}[X^2] - \mathbf{E}[Y^2] = \text{var}(X) - \text{var}(Y) = 0,$$

since X and Y were assumed to have the same variance.
$X + Y$ and $X - Y$ can be dependent or independent. Example for $X + Y$ and $X - Y$ being dependent: X and Y are the face up numbers corresponding to rolling two independent dice. Example for $X + Y$ and $X - Y$ being independent: X and Y are the "sum" and the "difference" of the face up numbers corresponding to rolling two independent dice.