Massachusetts Institute of Technology

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6.041/6.431: Probabilistic Systems Analysis

(Fall 2008)

Tutorial 7 Solutions [1]

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1. (a) Let N be the outcome of die. N is a discrete random variable taking values of 1,2, or 3 equally likely. So,

$$\begin{aligned}\mathbf{E}[N] &= \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 2 + \frac{1}{3}\cdot 3 \\ &= 2 \\ \text{var}(N) &= \frac{1}{3}(1-\mathbf{E}[N])^2 + \frac{1}{3}(2-\mathbf{E}[N])^2 + \frac{1}{3}(3-\mathbf{E}[N])^2 \\ &= \frac{2}{3}\end{aligned}$$

Let X be the result of spinning the wheel of fortune. It is a uniform random variable between $(0,1)$. Thus:

$$\begin{aligned}\mathbf{E}[X] &= \frac{1}{2} \\ \text{var}(X) &= \frac{1}{12}\end{aligned}$$

$Y = X_1 + \cdots + X_N$ where N is the outcome of die. Using properties listed on pg. 242 of the text, we get:

$$\begin{aligned}\mathbf{E}[Y] &= \mathbf{E}[X]\mathbf{E}[N] \\ &= 2\left(\frac{1}{2}\right) \\ &= 1\end{aligned}$$

(b) Similarly,

$$\begin{aligned}\text{var}(Y) &= \text{var}(X)\mathbf{E}[N] + \mathbf{E}[X]^2\text{var}(N) \\ &= \left(\frac{1}{12}\right)2 + \left(\frac{1}{2}\right)^2\frac{2}{3} \\ &= \frac{1}{6} + \frac{2}{12} \\ &= \frac{1}{3}\end{aligned}$$

2. The number of people who are smokers S, can be expressed as $S = X_1 + X_2 \ldots X_n$, where X_i is a bernoulli random variable with parameter f. Therefore, $F = \frac{S}{n} = \frac{X_1+X_2\ldots X_n}{n}$ has mean f and variance $f(1-f)/n$ (since $X_1, X_2 \ldots$ are independent.) According to Chebyshev inequality, we have

$$\mathbf{P}(|F-f| \geq d) \leq \frac{f(1-f)}{nd^2}$$

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a) We are given that $\frac{f(1-f)}{n_1 d^2} \leq (p = 0.05)$ for $n_1 \geq 50,000$. However, if we replace d with $d/2$, and if n_2 is the new number we need,

$$\frac{f(1-f)}{n_2 (d/2)^2} \leq (p = 0.05)$$

Therefore we need $n \geq 4n_1$ to maintain the inequality. Therefore $n_2 > 200,000$.

b) If we replace p with $0.5p$ and n_3, then we have

$$\frac{f(1-f)}{n_3 d^2} \leq (0.5p)$$

This is true if $n_3 \geq 2n_1$. Therefore, we need $n_3 > 100,000$.

3. Given:

$Z_n = \min(X_1, X_2, \ldots, X_n)$

To be convergent in probability, we need

$$\lim_{n \to \infty} \mathbf{P}(|Z_n - c| > \epsilon) = 0$$

for every $\epsilon > 0$. For our problem, let $c = 0$, since as $n \to \infty$, we expect our min result to be near 0.

$$\begin{aligned} \mathbf{P}(|Z_n - 0| > \epsilon) &= \mathbf{P}(Z_n > \epsilon) \\ &= \mathbf{P}(\min(X_1, X_2, \ldots, X_n) > \epsilon) \\ &= \mathbf{P}(X_1 > \epsilon) \cdot \mathbf{P}(X_2 > \epsilon) \cdot \cdots \cdot \mathbf{P}(X_n > \epsilon). \end{aligned}$$

Now, the probability of any single outcome of X being greater than ϵ is $1 - \epsilon$.

Thus,

$$\mathbf{P}(|Z_n - 0| > \epsilon) = (1 - \epsilon)^n.$$

So, as we take the limit

$$\lim_{n \to \infty} \mathbf{P}(|Z_n - 0| > \epsilon) = \lim_{n \to \infty} (1 - \epsilon)^n = 0,$$

and we find that indeed, $\{Z_n\}$ is convergent in probability to 0.