Tutorial 9: Solutions
November 6/7, 2008

1. (a) We need to know the number of fishing rods on and off the island. It is enough to know the number of fishing rods off the island. Therefore the states of our chain will be the number of fishing rods off the island. We will consider the state of the chain after a round trip to and from the island. This chain clearly has the Markov property, and thus it is easy to see that it is in fact a Markov chain.

   (b) This is a birth-death Markov chain, since the only states that are adjacent to S_i are states S_{i-1}, S_{i+1} for appropriate i. We first calculate the transition probabilities:

$$p_{ii} = \begin{cases} (1-p)^2 + p^2 & \text{for } 1 \le i \le n-1 \\ (1-p) & \text{for } i = 0 \\ 1-(1-p)p & \text{for } i = n \end{cases}$$

$$p_{i,i+1} = \begin{cases} (1-p)p & \text{for } 1 \le i < n \\ p & \text{for } i = 0 \end{cases}$$

$$p_{i,i-1} = \begin{cases} (1-p)p & \text{for } 1 \le i \\ 0 & \text{otherwise} \end{cases}$$

   For a birth-death Markov chain we have the following balance and normalization equations:

$$\pi_0 p_{01} = \pi_1 p_{10} \Rightarrow \pi_1 = \frac{\pi_0}{1-p}$$

$$\pi_n = \cdots = \pi_2 = \pi_1 = \frac{\pi_0}{1-p}$$

$$\sum_i \pi_i = \pi_0\left(1 + \frac{n}{1-p}\right) = 1.$$

   The steady-state probabilities are

$$\pi_0 = \frac{1-p}{n+1-p}$$

$$\pi_i = \frac{1}{n+1-p}, \text{for all } i > 0.$$

   (c) Let A denote the event that the weather is nice but the professor has no fishing rods with him. We are interested in the probability that the professor has no fishing rod during one of his round trips in the steady state. This will happen if: 1. The professor has either n rods at the house at the beginning of the round trip and the weather is bad on the way to the island and good on the way back; or 2. The professor has no rods at the house when leaving from the house to the island, and the weather is bad when coming from the island back the house.
   Then:

$$P(A) = P(A \mid \text{Markov process is in state } n) P(\text{Markov process is in state } n)$$

$$+P(A \mid \text{Markov process is in state } 0)P(\text{Markov process is in state } 0)$$

$$P(A) = \pi_n(1-p)p + \pi_0(p) = \frac{2p-2p^2}{n+1-p}.$$

d) If $n = 4$ we have:

$$P(A) = \frac{2p-2p^2}{5-p}$$

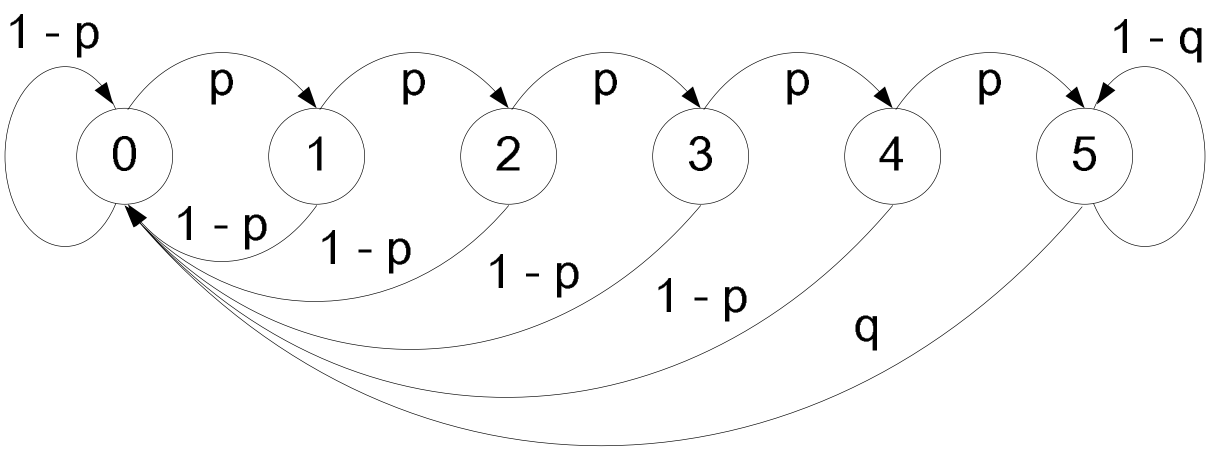
and we can maximize this by taking a derivative with respect to p and setting equal to zero. Solving, we find that the "optimal" value of p is:

$$p = \frac{10-\sqrt{80}}{2} = 5 - \sqrt{20} = 5 - 2\sqrt{5}.$$

2. Let us define the state space having 6 states, labeled 0,1,2,3,4 and 5, referring to the number of consecutive packet errors at a certain time. State 0 then corresponds to the initial state, or the state after a timeout, and state 5 corresponds to being in timeout. The transition probabilities are marked on Figure 1. As seen from the figure, this is a Markov chain with a single recurrent class, and there are no transient states. Then, the steady-state probabilities π_i will exist and be positive for $i = 0, 1, \ldots, 5$. Further, they satisfy the following balance equations:

$$\begin{aligned} p\pi_0 &= \pi_1 \\ p\pi_1 &= \pi_2 \\ p\pi_2 &= \pi_3 \\ p\pi_3 &= \pi_4 \\ p\pi_4 + (1-q)\pi_5 &= \pi_5 \end{aligned}$$

Using these, and $\sum_i \pi_i = 1$, we obtain $\pi_0 = \frac{1}{1+p+p^2+p^3+p^4+p^5/q}$, and the rest can be found from the equations above.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