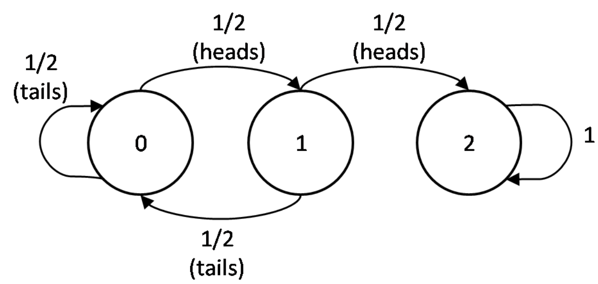
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2008)

Tutorial 10
November 13/14, 2008

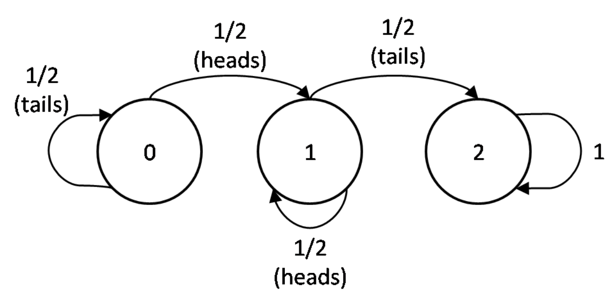
1. Consider a sequence of independent tosses of a fair coin. Let N_{HH} represent the number of tosses up to and including the first appearance of the sequence HH, and let N_{HT} represent the number of tosses up to and including the first appearance of the sequence HT. (For the sequence T H H H T H, for example, $N_{HH} = 3$ and $N_{HT} = 5$.)

   (a) Which of the following three possibilities would you expect is the most plausible?
      i. $E[N_{HH}] < E[N_{HT}]$
      ii. $E[N_{HH}] = E[N_{HT}]$
      iii. $E[N_{HH}] > E[N_{HT}]$

   (b) For N_{HH} make a 3-state Markov chain model with states k = 0, 1 and 2, where the chain starts in state 0 and resides there if the most recent toss was T, resides in state 1 if the most recent outcome was H not preceeded by H, and is in the trapping state 2 if the most recent two symbols were HH. (The symbol k represents how far the toss sequence has progressed towards the first appearance of HH.) Use this Markov chain (see below) to find $E[N_{HH}]$.



   (c) Create a similar 3-state Markov chain with states l = 0, 1 and 2 to represent N_{HT} . The chain starts in state 0 and resides there if the most recent outcome was T not preceeded by H, resides in state 1 if the most recent outcome was H, and resides in the trapping state 2 if the most recent outcomes were HT (see below). (The symbol l represents how far the toss sequence has progressed toward the first appearance of HT.) Find $E[N_{HT}]$ Compare your results to your initial answer to part (a). Explain in simple intuitive language what features of the chains lead to the outcome you found.



2. Sam and Pat are playing foosball. When they begin, the score is 0-0. To make things interesting, if the score ever becomes tied, it is instantly reset to 0-0. Starting from any score, the probability that Sam gets the next point is $\frac{1}{3}$.

(a) Suppose the game stops when one player's score reaches 2.

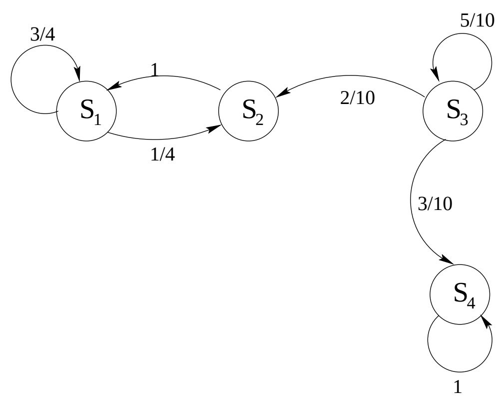
i. Draw an appropriate Markov chain that describes the game.

ii. Identify all transient, recurrent, and periodic states.

iii. Find $\mathbf{P}$(Pat wins).

(b) Now suppose instead that the game stops when a total of 3 points have been scored (note that this stopping condition does not explicitly depend on the score). The player with the most points when the game ends wins. Draw an appropriate Markov chain that describes the game.

3. Consider the Markov chain below. For all parts of this problem except (c), the process is in state 3 immediately before the first transition. Be sure to comment on any unusual results.



(a) Find the variance for J, the number of transitions up to and including the transition on which the process leaves state 3 for the last time.

(b) Find the expectation for K, the number of transitions up to and including the transition on which the process enters state 4 for the first time.

(c) Find for all values of i and j:

$$\lim_{t\to\infty} P(\text{state } t \text{ is } S_i|\text{initial state is } S_j)$$

(d) Find π_i for $i = 1, 2, \ldots, 4$, the probability that the process is in state i after 10^{10} transitions or explain why these probabilities can't be found.

(e) Given that the process never enters state 4, find the π_i's as defined in part (d) or explain why they can't be found