Tutorial 10: Solutions
November 13/14, 2008

1. Consider a sequence of independent tosses of a fair coin. Let N_{HH} represent the number of tosses up to and including the first appearance of the sequence HH, and let N_{HT} represent the number of tosses up to and including the first appearance of the sequence HT. (For the sequence T H H H T H, for example, $N_{HH} = 3$ and $N_{HT} = 5$.)

   (a) While intuitively, one might expect $E[N_{HH}] = E[N_{HT}]$ to be true, as we will find out in the following two sections, it is not.

   (b) Let t_0, t_1, and t_2 respectively be the expected time to absorption for states 0, 1, and 2. Then, we have a set of equations as follows:

$$\begin{aligned}
t_2 &= 0 \\
t_1 &= 1 + 1/2 \cdot t_2 + 1/2 \cdot t_0 \\
&= 1 + 1/2 \cdot t_0 \\
t_0 &= 1 + 1/2 \cdot t_0 + 1/2 \cdot t_1 \\
&= 1 + 1/2 \cdot t_0 + 1/2(1 + 1/2 \cdot t_0) \\
&= 1 + 2/4 \cdot t_0 + 1/2 + 1/4 \cdot t_0 \\
&= 6/4 + 3/4 \cdot t_0 \\
1/4 \cdot t_0 &= 6/4 \\
t_0 &= 6
\end{aligned}$$

   Thus, if we start at state 0, we expect 6 coin flips to the first time we see HH.

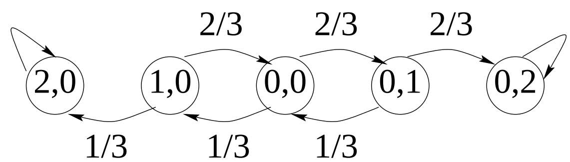
   (c) Similar to part (b), let u_0, u_1, and u_2 respectively be the expected time to absorption for states 0, 1, and 2. Then, we have a set of equations as follows:

$$\begin{aligned}
u_2 &= 0 \\
u_1 &= 1 + 1/2 \cdot u_2 + 1/2 \cdot u_1 \\
&= 1 + 1/2 \cdot u_1 \\
1/2 \cdot u_1 &= 1 \\
u_1 &= 2 \\
u_0 &= 1 + 1/2 \cdot u_0 + 1/2 \cdot u_1 \\
&= 1 + 1/2 \cdot u_0 + 1/2 \cdot (2) \\
&= 2 + 1/2 \cdot u_0 \\
1/2 \cdot u_0 &= 4/2 \\
u_0 &= 4
\end{aligned}$$

   Thus, if we start at state 0, we expect 4 coin flips to the first time we see HT.

   It takes on average, 2 fewer coin flips to first observe an HT as compared to an HH. In the chain for HH, after observing the first part of the desired sequence (i.e., a heads), observing a "mistake" (i.e., a tails) is resets the entire sequence back to state 0, while in the chain for HT, after observing the first part of the desired sequence (i.e., a heads), a "mistake" (i.e., another heads) only costs a self-transition to state 1.

2. (a) i. Denote by (x, y) the score of Sam and Pat respectively, a Markov chain that describes the game is



Note that the game ends when either state $(0, 2)$ or $(2, 0)$ is entered.

ii. Since we have a finite number of states, a state is recurrent if and only if it is accessible from all the states that are accessible from it, and therefore, states $(0, 0), (0, 1)$ and $(1, 0)$ are transient and states $(0, 2)$ and $(2, 0)$ are recurrent.

iii. The probability that Pat wins is the probability that we get absorbed to state $(0, 2)$. Setting up the equations, we solve for $a_{(1,0)}, a_{(0,0)}$ and $a_{(0,1)}$

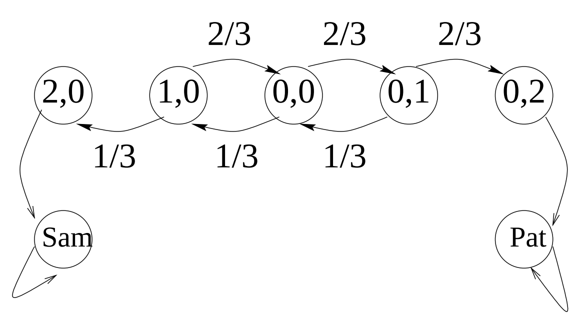
$$\begin{aligned} a_{(0,1)} &= \frac{2}{3} + \frac{1}{3} a_{(0,0)} \\ a_{(0,0)} &= \frac{2}{3} a_{(0,1)} + \frac{1}{3} a_{(1,0)} \\ a_{(1,0)} &= \frac{2}{3} a_{(0,0)} \end{aligned}$$

which yields that following

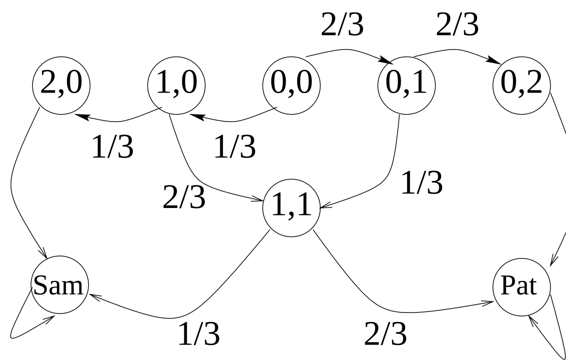
$$\begin{aligned} a_{(0,1)} &= \frac{14}{15} \\ a_{(0,0)} &= \frac{12}{15} \\ a_{(1,0)} &= \frac{8}{15} \end{aligned}$$

Therefore the probability of Pat winning is equal to $12/15 = 0.8$.

(b) The question can be interpreted in two ways: if the score is reset to 0-0 when the game is tied, then an appropriate Markov chain is



If the score is not reset, then the chain would look like



3. (a) The process is in state 3 immediately before the first transition. After leaving state 3 for the first time, the process cannot go back to state 3 again. Hence J, which represents the number of transitions up to and including the transition on which the process leaves state 3 for the first/last time is a geometric random variable with success probability equal to 0.5. The variance for J is given by:

$$\sigma_J^2 = \frac{1-p}{p^2} = 2$$

(b) There is a positive probability that we never enter state 4; i.e., $P(K < \infty) < 1$. Hence the expected value of K is ∞.

(c) The Markov chain has 2 different recurrent classes. The first recurrent class consists of states $\{1, 2\}$ and the second recurrent class consists of the state $\{4\}$. The probability of getting absorbed into the first recurrent class starting from the transient state 3 is,

$$\frac{2/10}{2/10 + 3/10} = \frac{2}{5}$$

which is the probability of transition to the first recurrent class given there is a change of state. Similarly, probability of absorption into the second recurrent class is $\frac{3}{5}$.

Now, we solve the balance equations within each recurrent class, which give us the probabilities conditioned on getting absorbed from state 3 to that recurrent class. The unconditional steady-state probabilities are found by weighing the conditional steady-state probabilities by the probability of absorption to the recurrent classes.

The first recurrent class is a birth-death process. We write the following equations and solve for the conditional probabilities, denoted by p_1 and p_2.

$$p_2 = \frac{p_1}{4}$$

$$p_1 + p_2 = 1$$

Solving these equations, we get $p_1 = \frac{4}{5}$, $p_2 = \frac{1}{5}$. For the second recurrent class, $p_4 = 1$.

Using this data, we find:

$$\lim_{t\to\infty} \mathbf{P}(S_i \text{ at step } t | \text{initial state is } S_j) = \lim_{n\to\infty} r_{j,i}(n)$$

$$\lim_{n\to\infty} r_{1,1}(n) = 4/5$$

$$\begin{aligned}
\lim_{n\to\infty} r_{2,1}(n) &= 4/5 \\
\lim_{n\to\infty} r_{3,1}(n) &= 2/5 \cdot 4/5 = 8/25 \\
\lim_{n\to\infty} r_{1,2}(n) &= 1/5 \\
\lim_{n\to\infty} r_{2,2}(n) &= 1/5 \\
\lim_{n\to\infty} r_{3,2}(n) &= 2/5 \cdot 1/5 = 2/25 \\
\lim_{n\to\infty} r_{3,4}(n) &= 3/5 \\
\lim_{n\to\infty} r_{4,4}(n) &= 1 \\
\text{All other cases} &= 0
\end{aligned}$$

(d) There are two ways to interpret this problem. If you assumed that the process started in state 3, then the probabilities that the process is in state i after 10^{10} transitions are the relevant probabilities from part (c).
However, if you assumed that π_i denoted steady state probabililties, then these probabilities do not exist. From part (c), we know that

$$\lim_{n\to\infty} r_{ij}(n) \text{ is not independent of } i\text{, e.g } \lim_{n\to\infty} r_{1,1}(n) = 4/5, \text{ but } \lim_{n\to\infty} r_{4,1}(n) = 0$$

Thus, steady state probabilities do not exist.

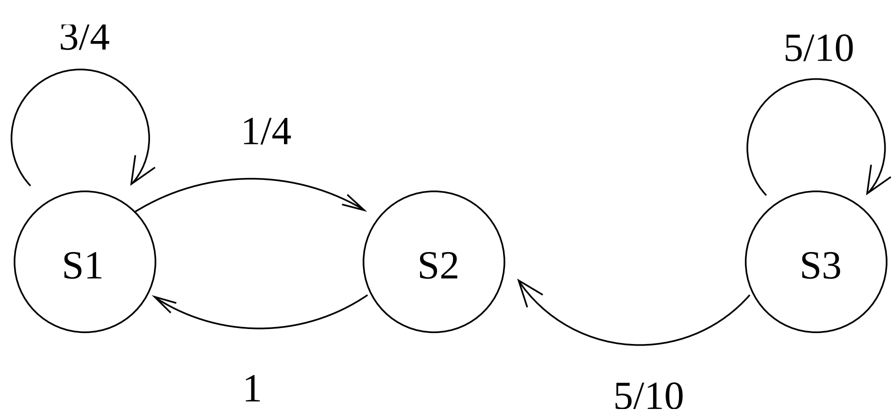
(e) The given conditional event, that the process never enters state 4, changes the absorption probabilities to the recurrent classes. The probability of getting absorbed to the first recurrent class is 1, and to the second recurrent class is 0. Hence, the steady state probabilities now exist and they are given by,

$$\pi_1 = \frac{4}{5} \cdot 1 = \frac{4}{5}$$

$$\pi_2 = \frac{1}{5} \cdot 1 = \frac{1}{5}$$

$$\pi_3 = \pi_4 = 0$$

For pedagogical purposes, let us actually draw what the new Markov chain would look like, given the event that the process never enters state 4. The resulting chain is shown below. Let us see how we came up with these transition probabilities.



We need to be careful when rescaling the new transition probabilities. First of all, it is clear that the probabilities within the recurrent class $\{S1, S2\}$ doesn't get affected. We also note that the self loop transition probability of the transient state $S3$ doesn't get changed either. (this would be true for any other transient state if they were present.)

To see that the self loop probability $p_{3,3}$ doesn't get changed, we condition on the event that we eventually enter S2. Let's call the new self loop probability, $q_{3,3}$.
Then,

$$q_{3,3} = \mathbf{P}(X_1 = S3 | \, absorbed\ into\ 2, X_0 = S3) = \frac{p_{3,3} * \mathbf{P}(absorbed\ into\ 2\ | X_1 = S3,\ X_0 = S3)}{\mathbf{P}(absorbed\ into\ 2\ |\ X_0 = S3)}$$

$$= \frac{p_{3,3} * a_{3,2}}{a_{3,2}} = p_{3,3} = \frac{5}{10}$$

The important thing to take away from this example is that, when doing problems of this sort, (i.e given we do/don't enter a particular set of recurrent classes), it is neccessary to rescale the transition probabilities of the new chain, coming out of ALL the transient states. In other words, to find each of the new transition probabilities, we condition on the given event, that we do or do not enter particular recurrent classes.