Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2008)

Tutorial 11
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1. We wish to estimate the probability of heads of a biased coin, which we denote θ. We model θ as the value of a random variable Θ distributed over $[0,1]$ according to the PDF

$$f_\Theta(\theta) = 2 - 4\left|\frac{1}{2} - \theta\right|, \qquad \theta \in [0,1]$$

   Find the MAP estimate of Θ, assuming that n independent coin tosses resulted in k heads and $n-k$ tails.

2. The probability of heads of a given coin is known to be either q_0 (Hypothesis H_0) or q_1 (Hypothesis H_1). We toss the coin repeated and independently, and record the number of heads before a tail is observed for the first time. We assume that $0 < q_0 < q_1 < 1$, and that we are given prior probabilities $\mathbf{P}(H_0)$ and $\mathbf{P}(H_1)$. For parts (a) and (b), we also assume that $\mathbf{P}(H_0) = \mathbf{P}(H_1) = 1/2$.

   (a) Calculate the probability that hypothesis H_1 is true, given that there were exactly k heads before the first tail.

   (b) Consider the decision rule that decides in favor of hypothesis H_1 if $k \geq k^*$, where k^* is some nonnegative integer, and decides in favor of hypothesis H_0 otherwise. Give a formula for the probability of error in terms of k^*, q_0, and q_1. For what value of k^* is the probability of error minimized? Is there another type of decision rule that would lead to an even lower probability of error?

   (c) Assume that $q_0 = 0.3$, $q_1 = 0.7$, and $\mathbf{P}(H_1) > 0.7$. How does the optimal choice of k^* (the one that minimizes the probability of error) change as $\mathbf{P}(H_1)$ increases from 0.7 to 1.0?