Problem 5.33 Dimensionless form of the well-depth analysis

Even the messiest results are cleaner and have lower entropy in dimensionless form. The four quantities \( h, g, T, \) and \( c_s \) produce two independent dimensionless groups (Section 2.4.1). An intuitively reasonable pair are

\[
\bar{h} \equiv \frac{h}{gT^2} \quad \text{and} \quad \bar{T} \equiv \frac{gT}{c_s}.
\]

(5.40)

a. What is a physical interpretation of \( \bar{T} \)?

b. With two groups, the general dimensionless form is \( \bar{h} = f(\bar{T}) \). What is \( \bar{h} \) in the easy case \( \bar{T} \to 0 \)?

c. Rewrite the quadratic-formula solution

\[
h = \left( \frac{-\sqrt{2/g} + \sqrt{2/g + 4T/c_s}}{2/c_s} \right)^2
\]

(5.41)

as \( \bar{h} = f(\bar{T}) \). Then check that \( f(\bar{T}) \) behaves correctly in the easy case \( \bar{T} \to 0 \).

Problem 5.34 Spacetime diagram of the well depth

How does the spacetime diagram [44] illustrate the successive approximation of the well depth? On the diagram, mark \( h_0 \) (the zeroth approximation to the depth), \( h_1 \), and the exact depth \( h \). Mark \( t_0 \), the zeroth approximation to the free-fall time. Why are portions of the rock and sound-wavefront curves dotted? How would you redraw the diagram if the speed of sound doubled? If \( g \) doubled?

5.5 Daunting trigonometric integral

The final example of taking out the big part is to estimate a daunting trigonometric integral that I learned as an undergraduate. My classmates and I spent many late nights in the physics library solving homework problems; the graduate students, doing the same for their courses, would regale us with their favorite mathematics and physics problems.

The integral appeared on the mathematical-preliminaries exam to enter the Landau Institute for Theoretical Physics in the former USSR. The problem is to evaluate

\[
\int_{-\pi/2}^{\pi/2} (\cos t)^{100} \, dt
\]

(5.42)
5.5 Daunting trigonometric integral

to within 5% in less than 5 min without using a calculator or computer! That \((\cos t)^{100}\) looks frightening. Most trigonometric identities do not help. The usually helpful identity \((\cos t)^2 = (\cos 2t - 1)/2\) produces only

\[
(\cos t)^{100} = \left( \frac{\cos 2t - 1}{2} \right)^{50},
\]

which becomes a trigonometric monster upon expanding the 50th power. A clue pointing to a simpler method is that 5% accuracy is sufficient—so, find the big part! The integrand is largest when \(t\) is near zero. There, \(\cos t \approx 1 - t^2/2\) (Problem 5.20), so the integrand is roughly

\[
(\cos t)^{100} \approx \left( 1 - \frac{t^2}{2} \right)^{100}.
\]

It has the familiar form \((1 + z)^n\), with fractional change \(z = -t^2/2\) and exponent \(n = 100\). When \(t\) is small, \(z = -t^2/2\) is tiny, so \((1 + z)^n\) may be approximated using the results of Section 5.3.4:

\[
(1 + z)^n \approx \begin{cases} 1 + nz & (z \ll 1 \text{ and } nz \ll 1) \\ e^{nz} & (z \ll 1 \text{ and } nz \text{ unrestricted}) \end{cases}.
\]

Because the exponent \(n\) is large, \(nz\) can be large even when \(t\) and \(z\) are small. Therefore, the safest approximation is \((1 + z)^n \approx e^{nz}\); then

\[
(\cos t)^{100} \approx \left( 1 - \frac{t^2}{2} \right)^{100} \approx e^{-50t^2}.
\]

A cosine raised to a high power becomes a Gaussian! As a check on this surprising conclusion, computer-generated plots of \((\cos t)^n\) for \(n = 1 \ldots 5\) show a Gaussian bell shape taking form as \(n\) increases.

Even with this graphical evidence, replacing \((\cos t)^{100}\) by a Gaussian is a bit suspicious. In the original integral, \(t\) ranges from \(-\pi/2\) to \(\pi/2\), and these endpoints are far outside the region where \(\cos t \approx 1 - t^2/2\) is an accurate approximation. Fortunately, this issue contributes only a tiny error (Problem 5.35). Ignoring this error turns the original integral into a Gaussian integral with finite limits:

\[
\int_{-\pi/2}^{\pi/2} (\cos t)^{100} \, dt \approx \int_{-\pi/2}^{\pi/2} e^{-50t^2} \, dt.
\]
Unfortunately, with finite limits the integral has no closed form. But extending the limits to infinity produces a closed form while contributing almost no error (Problem 5.36). The approximation chain is now

\[
\int_{-\pi/2}^{\pi/2} (\cos t)^{100} \, dt \approx \int_{-\pi/2}^{\pi/2} e^{-50t^2} \, dt \approx \int_{-\infty}^{\infty} e^{-50t^2} \, dt. \tag{5.48}
\]

**Problem 5.35 Using the original limits**

The approximation \(\cos t \approx 1 - t^2/2\) requires that \(t\) be small. Why doesn’t using the approximation outside the small-\(t\) range contribute a significant error?

**Problem 5.36 Extending the limits**

Why doesn’t extending the integration limits from \(\pm \pi/2\) to \(\pm \infty\) contribute a significant error?

The last integral is an old friend (Section 2.1): \(\int_{-\infty}^{\infty} e^{-\alpha t^2} \, dt = \sqrt{\pi/\alpha}\). With \(\alpha = 50\), the integral becomes \(\sqrt{\pi/50}\). Conveniently, 50 is roughly \(16\pi\), so the square root—and our 5% estimate—is roughly 0.25.

For comparison, the exact integral is (Problem 5.41)

\[
\int_{-\pi/2}^{\pi/2} (\cos t)^n \, dt = 2^{-n} \left( \binom{n}{n/2} \right) \pi. \tag{5.49}
\]

When \(n = 100\), the binomial coefficient and power of two produce

\[
\frac{12611418068195524166851562157}{158456325028528675187087900672} \pi \approx 0.25003696348037. \tag{5.50}
\]

Our 5-minute, within-5% estimate of 0.25 is accurate to almost 0.01%!

**Problem 5.37 Sketching the approximations**

Plot \((\cos t)^{100}\) and its two approximations \(e^{-50t^2}\) and \(1 - 50t^2\).

**Problem 5.38 Simplest approximation**

Use the linear fractional-change approximation \((1 - t^2/2)^{100} \approx 1 - 50t^2\) to approximate the integrand; then integrate it over the range where \(1 - 50t^2\) is positive. How close is the result of this 1-minute method to the exact value 0.2500…?

**Problem 5.39 Huge exponent**

Estimate

\[
\int_{-\pi/2}^{\pi/2} (\cos t)^{10000} \, dt. \tag{5.51}
\]