

Approximating logarithms using musical intervals

Semitones	Interval	Ratio	Exact Value
2	M2	9/8	1.122
3	m3	6/5	1.1885
4	M3	5/4	1.259
5	P4	4/3	1.3335
6	d5	$\sqrt{2}$	1.4125
7	P5	3/2	1.496
8	m6 = P8 - M3	8/5	1.585
9	M6 = P8 - m3	5/3	1.679
10	P5 + m3	9/5	1.7783
	2 · P4	16/9	1.7783
11		17/9	1.8836
12	P8	2	1.9953
17.4		e	2.718
19	P8 + P5	3	2.9854
24	2 · P8	4	3.981
28	2 · P8 + M3	5	5.012
31	2 · P8 + P5	6	5.9566
34	3 · P8 - M2	$\frac{64}{9} \approx 7$	7.080
36	3 · P8	8	7.943
38	2 · (P8 + P5)	9	8.913
40	3 · P8 + M3	10	10.

KEY		
Symbol	Interval	Notes
M2	Major 2nd	C-D
m3	Minor 3rd	C-E \flat
M3	Major 3rd	C-E
P4	Perfect 4th	C-F
d5	Diminished 5th	C-G \flat
P5	Perfect 5th	C-G
m6	Minor 6th	C-A \flat
M6	Major 6th	C-A

The starting point is $2^{10} \approx 10^3$, or $2^{1/12} \approx 10^{1/40}$. By chance $2^{1/12}$ is the semitone frequency ratio on the equal-tempered scale. Since we know what Pythagorean ratios the equal-tempered intervals are supposed to approximate, we can approximate logarithms to the base $2^{1/12}$, and thereby approximate logarithms to the base $10^{1/40}$, which gives us twice the number of decibels. The ratio column indicates the ratios for perfect Pythagorean intervals, and the exact value column shows $10^{\text{semitones}/40}$, to show the accuracy of the method. Note that 10 semitones has two possible breakdowns into intervals, as P5 + m3 or 2 · P4. The second is much more accurate, because in the equal-tempered scale, the perfect intervals come out almost exactly right, at the cost of some error in the major and minor intervals.

To use the table to compute $\log_{10} x$, find x as a product of ratios, add the number of semitones for the ratios, and divide by 40 (divide by 2 to get dB). To calculate 10^x , multiply x by 40, find that value in the semitones column, and read off the corresponding ratio. From a few basic Pythagorean ratios and number of semitones, most of the table is easy to figure out. The most important to remember one is the fifth: 7 semitones corresponds to 3/2. For example, from the fifth we can compute the frequency ratio for a fourth (5 semitones). The two intervals together make an octave, so the product of their frequency ratios is 2. This means 5 semitones corresponds to 4/3. Many other entries can be worked out similarly.

Some examples (arrows point from the real to the log world):

$$2 \rightarrow 1 \text{ octave} = 12 \text{ semitones} = 6 \text{ dB} = 0.3 \text{ decades.}$$

$$\left(\frac{4}{3}\right)^{10} \rightarrow 10 \cdot \text{P4} = 50 \text{ semitones} = 40 \text{ semitones} + 2 \cdot \text{P4} \leftarrow 10 \cdot \frac{16}{9} = 17.78 \text{ (exact 17.76).}$$

$$5 = \frac{5}{4} \cdot 2 \cdot 2 \rightarrow \text{M3} + 2 \cdot \text{P8} = 28 \text{ semitones} = \frac{28}{40} \text{ or } 0.7 \text{ decades (14 dB).}$$

$$3^{10} \rightarrow 10 \cdot (\text{P8} + \text{P5}) = 190 \text{ semitones} = 200 - 2 \cdot \text{P4} \leftarrow 10^{200/40} \cdot \frac{9}{16} = 56250 \text{ (exact 59049).}$$

$$e^{10} \rightarrow 10 \cdot 17.4 \text{ semitones} = 174 \text{ semitones} = 160 + 12 + 2 \text{ semitones} \leftarrow 10^4 \cdot 2 \cdot \frac{9}{8} = 22500.$$

(This method is due to the statistician I. J. Good, who credits his father.)