Homework 3

Do the following warmups and problems. Submit your answers, including the short explanation, online by 10pm on Wednesday, 3 Mar 2010.

Open universe: Collaboration, notes, and other sources of information are encouraged. However, avoid looking up answers to the problem, or to subproblems, until you solve the problem or have tried hard. This policy helps you learn the most from the problems.

Homework is graded with a light touch: P (made a decent effort), D (made an indecent effort), or F (did not make an effort).

Warmups

1. Fuel efficiency of a 747
   Use the cost of a plane ticket to estimate the fuel efficiency of a 747, in passenger–miles per gallon (passenger–mpg).

   \[ 10 \pm \square \text{ passenger–mpg} \quad \text{or} \quad 10 \ldots \square \text{ passenger–mpg} \]

2. High winds
   At roughly what wind speed is the force on your body from the wind approximately equal to your weight?

   \[ \pm \square \text{ m} \text{s}^{-1} \quad \text{or} \quad \square \ldots \square \text{ m} \text{s}^{-1} \]

3. Daunting integral
   Evaluate
   \[
   \int_{-\infty}^{\infty} \frac{x^3}{1 + 7x^2 + 18x^6} \, dx. \tag{1}
   \]

   \[ \pm \square \quad \text{or} \quad \square \ldots \square \]

Problems

4. Solitaire
   You start with the numbers 3, 4, and 5. At each move, you choose any two of the three numbers – call the choices \(a\) and \(b\) – and replace them with \(0.8a - 0.6b\) and \(0.6a + 0.8b\). The goal is to reach 4, 4, 4. Can you do it? If yes, give a move sequence; if no, show that you cannot.
5. Maximizing a polynomial

Use symmetry to find the maximum value of \(6x - x^2\).

\[ \pm \ \text{or} \ \ldots \]

6. Tiling a mouse-eaten chessboard

An \(8 \times 8\) chessboard gets two diagonally opposite corners eaten away by a mouse. You have dominoes, each \(2 \times 1\) in shape – i.e. each covers two adjacent squares. Can you tile the mouse-eaten chessboard with these dominoes? In other words, can you lay down the dominoes to cover every square exactly once (no empty squares and no overlaps)?

\[ \text{yes} \]
\[ \text{no} \]

Optional!

7. Symmetry for second-order systems

This problem analyzes the frequency of maximum gain for an LRC circuit or, equivalently, for a damped spring–mass system. The gain of such a system is the ratio of the input amplitude to the output amplitude as a function of frequency.

If the output voltage is measured across the resistor, and you drive the circuit with a voltage oscillating at frequency \(\omega\), the gain is (in a suitable system of units):

\[
G(\omega) = \frac{j\omega}{1 + j\omega/Q - \omega^2},
\]

where \(j = \sqrt{-1}\) and \(Q\) is quality factor, a dimensionless measure of the damping. Do not worry if you do not know where that gain formula comes from. The purpose of this problem is not its origin, but rather using symmetry to maximize its magnitude.

The magnitude of the gain is

\[
|G(\omega)| = \frac{\omega}{\sqrt{(1 - \omega^2)^2 + \omega^2/Q^2}}.
\]

Find a variable substitution (a symmetry operation) \(\omega_{\text{new}} = f(\omega)\) that turns \(|G(\omega)|\) into \(|H(\omega_{\text{new}})|\) such that \(G\) and \(H\) are the same function (i.e. they have the same structure but with \(\omega\) in \(G\) replaced by \(\omega_{\text{new}}\) in \(H\)). Use the form of that symmetry operation to maximize \(|G(\omega)|\) without using calculus.

8. Inertia tensor

[For those who know about inertia tensors.] Here is the inertia tensor (the generalization of moment of inertia) of a particular object, calculated in a lousy coordinate system:

\[
\begin{pmatrix}
4 & 0 & 0 \\
0 & 5 & 4 \\
0 & 4 & 5 \\
\end{pmatrix}
\]

Change coordinate systems to a set of principal axes. In other words, write the inertia tensor as
\[
\begin{pmatrix}
I_{xx} & 0 & 0 \\
0 & I_{yy} & 0 \\
0 & 0 & I_{zz}
\end{pmatrix}
\]

and give the values of \(I_{xx}, I_{yy},\) and \(I_{zz}\). \textit{Hint:} What properties of a matrix are invariant when changing coordinate systems?

9. \textbf{Resistive grid}

In an infinite grid of 1-ohm resistors, what is the resistance measured across one resistor?

To measure resistance, an ohmmeter injects a current \(I\) at one terminal (for simplicity, say \(I = 1\, \text{A}\)), removes the same current from the other terminal, and measures the resulting voltage difference \(V\) between the terminals. The resistance is \(R = V/I\).

\textit{Hint:} Use symmetry. But it’s still a hard problem!