6.055J/2.038J (Spring 2010)

Solution set 4

Submit your answers and explanations online by ***10pm on Wednesday, 10 Mar 2010****.*

Open universe: *Collaboration, notes, and other sources of information are* ***encouraged.*** *However, avoid looking up answers to the problem, or to subproblems, until you solve the problem or have tried hard. This policy helps you learn the most from the problems.*

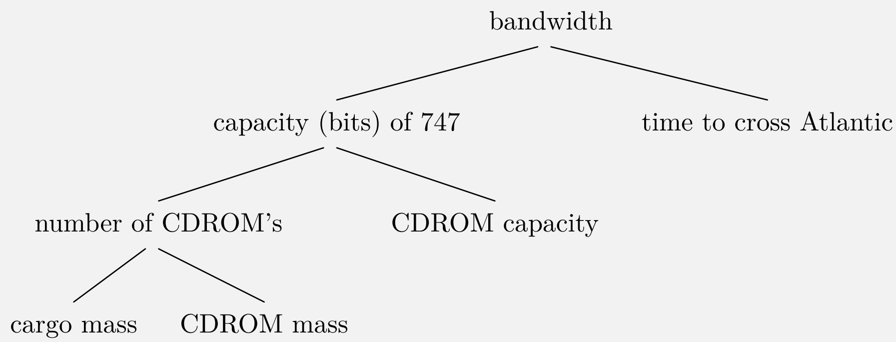
Homework is graded with a light touch: P (made a decent effort), D (made an indecent effort), or F (did not make an effort).

Problem 1 Bandwidth

To keep your divide-and-conquer muscles strong, here is an exercise from lecture: Estimate the bandwidth of a 747 crossing the Atlantic filled with CDROMs.

$10^{\square} \pm \square$ bits/s ***or*** $10^{\square} \ldots \square$ bits/s

Divide and conquer! Here's a tree on which to fill values:



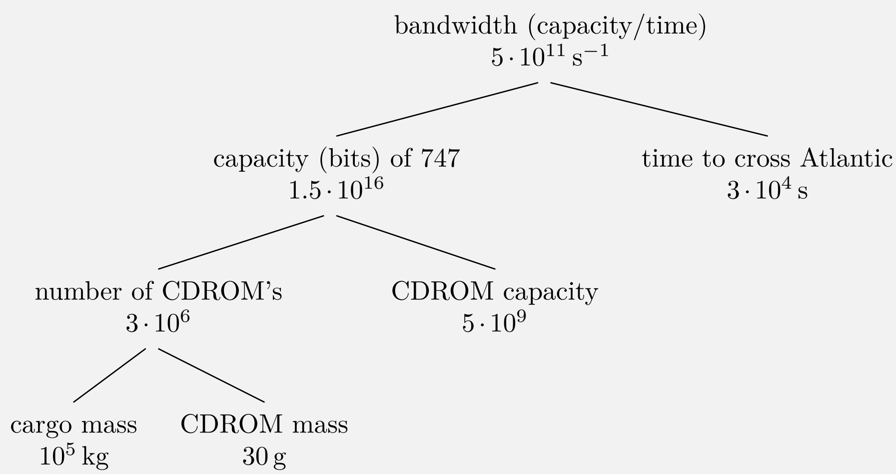
First I estimate the cargo mass. A 747 can easily carry about 400 people, each person having a mass (with luggage) of, say 140 kg. The total mass is

$$m \sim 400 \times 140\,\text{kg} \sim 6 \cdot 10^4\,\text{kg}.$$

A special cargo plane, with no seats or other frills for passengers, probably can carry 10^5 kg.

Here are the other estimates. A CDROM's mass is perhaps one ounce or 30 g. So the number of CDROM's is $3 \cdot 10^6$. The capacity of a CDROM is 600 MB or about $5 \cdot 10^9$ bits. The time to cross the Atlantic is about 8 hours or $3 \cdot 10^4$ s.

Now propagate the values toward the root of the tree:



The bandwidth is 0.5 terabits per second or $10^{11.5}$ bits/second.

Despite the large bandwidth offered by a 747 carrying CDROM's (not to mention DVDROM's), trans-Atlantic Internet connections go instead via undersea fiber-optic cables. Low latency is important!

Problem 2 Gravity versus radius

Assume that planets are uniform spheres. How does g, the gravitational acceleration at the surface, depend on the planet's radius R? In other words, what is the exponent n in

$$g \propto R^n? \tag{1}$$

$\square \pm \square$ ***or*** $\square \ldots \square$

The gravitational force (the weight) on an object of mass m is GMm/R^2, where G is Newton's constant, and M is the moon's mass. Thus the gravitational acceleration g is GM/R^2. But the mass M is proportional to R^3, so $g \propto R^1$. In other words, $n = 1$.

Problem 3 Gravity on the moon

The radius of the moon is one-fourth the radius of the earth. Use the result of **Problem 2** to predict the ratio $g_{\text{moon}}/g_{\text{earth}}$. In reality, $g_{\text{moon}}/g_{\text{earth}}$ is roughly one-sixth. How might you explain any discrepancy between the predicted and actual ratio?

The ratio $g_{\text{moon}}/g_{\text{earth}}$ should be proportional to the ratio of radii $R_{\text{moon}}/R_{\text{earth}}$, namely 1/4. The actual ratio is lower because of an effect neglected in the analysis of **Problem 2**: the differing density. When that effect is included, then the mass M is ρR^3 (except for a constant), so

$$g \sim \frac{G\rho R^3}{R^2} \propto \rho R. \tag{2}$$

If $\rho_{\text{moon}}/\rho_{\text{earth}}$ is 2/3, that reduction in concert with the radii ratio would explain the factor of 6 difference in g.

Moon rock, which is less dense than the average earth rock, is comparable in density to rock in the earth's crust. This equivalence suggests that the moon was once a piece of the earth's crust that got scooped out probably by a large meteor impact.

Problem 4 Minimum power

In the readings we estimated the flight speed that minimizes energy consumption. Call that speed v_E. We could also have estimated v_P, the speed that minimizes power consumption. What is the ratio v_P/v_E?

[ ] ± [ ] *or* [ ] . . . [ ]

The zillions of constants (such as ρ) clutter the analysis without changing the result. So I'll simplify the problem by using a system of units where all the constants are 1. Then the energy is

$$E \sim v^2 + \frac{1}{v^2},$$

where the first term is from drag and the second term is from lift. The power is energy per time, and time is inversely proportional to v, so $P \propto Ev$ and

$$P \sim v^3 + \frac{1}{v}.$$

The first term is the steep v^3 dependence of drag power on velocity (which we used to estimate the world-record cycling and swimming speeds). The energy expression is unchanged when $v \to 1/v$, so it has a minimum at $v_E = 1$.

To minimize the power, use calculus (ask me if you are curious about calculus-free ways to minimize it):

$$\frac{dP}{dv} \sim 3v^2 - \frac{1}{v^2} = 0,$$

therefore $v_P = 3^{-1/4}$ (roughly 3/4), which is also the ratio v_P/v_E.

So the minimum-power speed is about 25% less than the minimum-energy speed. That result makes sense. Drag power grows very fast as v increases – much faster than lift power decreases – so it's worth reducing the speed a little to reduce the drag a lot.

If you don't believe the simplification that I used of setting all constants to 1 – and it is not immediately obvious that it should work – then try using this general form:

$$E \sim Av^2 + \frac{B}{v^2},$$

where A and B are constants. You'll find that v_E and v_P each contain the same function of A and B and that this function disappears from the ratio v_P/v_E.

Problem 5 Highway vs city driving

Here is a measure of the importance of drag for a car moving at speed v for a distance d:

$$\frac{E_{\text{drag}}}{E_{\text{kinetic}}} \sim \frac{\rho v^2 A d}{m_{\text{car}} v^2}.$$

This ratio is equivalent to the ratio

$$\frac{\text{mass of the air displaced}}{\text{mass of the car}}$$

and to the ratio

$$\frac{\rho_{\text{air}}}{\rho_{\text{car}}} \times \frac{d}{l_{\text{car}}},$$

where ρ_{car} is the density of the car (its mass divided by its volume) and l_{car} is the length of the car.

Make estimates for a typical car and find the distance d at which the ratio becomes significant (say, roughly 1).

$10^{\square} \pm \square$ m ***or*** $10^{\square} \ldots \square$ m

To include in the explanation box: How does the distance compare with the distance between exits on the highway and between stop signs or stoplights on city streets? What therefore are the main mechanisms of energy loss in city and in highway driving?

A typical car has mass $m_{\text{car}} \sim 10^3$ kg, cross-sectional area $A \sim 2\,\text{m} \times 1.5\,\text{m} = 3\,\text{m}^2$, and length $l_{\text{car}} \sim 4$ m. So

$$\rho_{\text{car}} \sim \frac{m_{\text{car}}}{A l_{\text{car}}} \sim \frac{10^3\,\text{kg}}{3\,\text{m}^2 \times 4\,\text{m}} \sim 10^2\,\text{kg}\,\text{m}^{-3}.$$

Since $\rho_{\text{car}}/\rho_{\text{air}} \sim 100$, the ratio

$$\frac{\rho_{\text{air}}}{\rho_{\text{car}}} \frac{d}{l_{\text{car}}}$$

becomes 1 when $d/l_{\text{car}} \sim 100$, so $d \sim 400$ m.

This distance d is significantly farther than the distance between stop signs or stoplights on city streets. In Manhattan, for example, 20 east–west blocks are one mile, giving a spacing of approximately 80 m. So air resistance is not a significant loss in city driving. Instead the loss comes from engine friction, rolling resistance, and (mostly) braking.

However, the distance d is comparable to the exit spacing on urban highways. So when you drive on the highway for even a few exit distances, air resistance is a significant loss.

Interestingly, highway fuel efficiencies are higher than city fuel efficiencies, even though drag gets worse at the higher, highway speeds, and presumably engine friction and rolling resistance also get worse at higher speeds. Only one loss mechanism, braking, is less prevalent in highway than in city driving. Therefore, braking must be a significant loss in city driving. Regenerative braking, used in some hybrid or electric cars, would therefore significantly improve fuel efficiency in city driving.

Problem 6 Mountains

Here are the heights of the tallest mountains on Mars and Earth.

Mars	27 km	(Mount Olympus)
Earth	9 km	(Mount Everest)

Predict the height of the tallest mountain on Venus.

$10^{\square} \pm \square$ km *or* $10^{\square} \ldots \square$ km

To include in the explanation box: Then check your prediction in a table of astronomical data (or online).

One pattern is that the larger planet (earth) has the smaller mountain. Large planets presumably have stronger gravitational fields at their surface, which keeps the mountains closer to the ground. The derivation in lecture on mountain heights dropped the dependence on g because we looked only at mountains on earth where all mountains share the value of g.

The same derivation can be repeated but retaining g. The weight of a mountain of size l is $W \propto gl^3$, so the pressure at the base is $p \propto gl^3/l^2 \sim gl$. When the pressure p exceeds the maximum pressure that rock can support, the mountain can no longer grow upward. This criterion is equivalent to holding gl constant. Therefore,

$$l \propto g^{-1}.$$

Here are the gravitational field strengths on the three planets:

a. Mars: $3.7\,\mathrm{m\,s^{-2}}$

b. earth: $10\,\mathrm{m\,s^{-2}}$

c. Venus: $8.9\,\mathrm{m\,s^{-2}}$

The product gl for each planet should be the same. That hypothesis works for Mars and earth:

a. Mars: $10^5\,\mathrm{m^2\,s^{-2}}$

b. earth: $0.9 \cdot 10^5\,\mathrm{m^2\,s^{-2}}$

If Venus follows the predicted scaling, then gl should be roughly $10^5\,\mathrm{m^2\,s^{-2}}$ with $g \sim 8.9\,\mathrm{m\,s^{-2}}$. Therefore l should be roughly 11 km. Indeed, the tallest mountain on Venus, which is Maxwell Montes, has just that height. Scaling triumphs!

Here is a fun question: Why aren't mountains on the moon 60 km tall (the Moon's surface gravity is about one-sixth of earth's surface gravity, as analyzed in **Problem 3**)?

Problem 7 Raindrop speed

Use the drag-force results from the readings to estimate the terminal speed of a typical raindrop (diameter of about 0.5 cm).

$10^{\square} \pm \square$ $\mathrm{m\,s^{-1}}$ ***or*** $10^{\square} \ldots \square$ $\mathrm{m\,s^{-1}}$

To include in the explanation box: How could you check this result?

The weight of the raindrop is the density times the volume times g:

$$W \sim \rho r^3 g,$$

where I neglect dimensionless factors such as $4\pi/3$.

At terminal velocity, the weight equals the drag. The drag is

$$F \sim \rho_{\text{air}} v^2 A \sim \rho_{\text{air}} v^2 r^2.$$

Equating the weight to the drag gives an equation for v and r:

$$\rho_{\text{air}} v^2 r^2 \sim \rho r^3 g,$$

so $v \propto r^{1/2}$.

Bigger raindrops fall faster but – because of the square root – not much faster.

With the g and the densities, the terminal velocity is

$$v \sim \sqrt{\frac{\rho}{\rho_{\text{air}}} gr}.$$

A typical raindrop has a diameter of maybe 5 or 6 mm, so $r \sim 3\,\text{mm}$. Since the density ratio between water and air is roughly 1000,

$$v \sim \sqrt{1000 \times 10\,\mathrm{m\,s^{-2}} \times 3 \cdot 10^{-3}\,\mathrm{m}} \sim 5\,\mathrm{m\,s^{-1}}.$$

First convert the speed into a more familiar value: 11 mph (miles per hour). If one drives at a speed v_{car}, then raindrops appear to move at an angle $\arctan(v_{\text{car}}/v)$. When $v_{\text{car}} = v$, the drops come at a 45° angle. So one way to measure the terminal speed is to drive in a rainstorm, slowly accelerating while the passenger (not the driver!) says when the drops hit at a 45° angle.

You could also run in a rainstorm and note the speed at which a small umbrella has to held at 45° to keep you perfectly dry.

Problem 8 Cruising speed versus air density

For geometrically similar animals (same shape and composition but different size), how does the minimum-energy speed v depend on air density ρ? In other words, what is the exponent β in $v \propto \rho^{\beta}$?

[ ] ± [ ] *or* [ ] . . . [ ]

From the lecture notes,

$$Mg \sim C^{1/2} \rho v^2 L^2,$$

where C is the modified drag coefficient. So

$$v \sim \left(\frac{Mg}{C^{1/2} \rho L^2} \right)^{1/2}.$$

The only dependence on ρ is the ρ itself in the denominator, leaving

$$v \propto \rho^{-1/2}$$

and $\beta = 1/2$.

The inverse relationship between the speed and density explains why planes fly at a high altitude. The energy consumption at the minimum-energy speed is proportional to the drag force, which is proportional to ρv^2. Because $v \propto \rho^{-1/2}$, the powers of ρ cancel in the energy consumption; in other words, the energy consumption (at the minimum-energy speed for that ρ) is independent of ρ. By flying high, where ρ is low, planes can fly faster without increasing their energy consumption.

Problem 9 Cruising speed versus mass

For geometrically similar animals (same shape and composition but different size), how does the minimum-energy speed v depend on mass M? In other words, what is the exponent β in $v \propto M^{\beta}$?

☐ ± ☐ ***or*** ☐ . . . ☐

Again from the lecture notes,

$$Mg \sim C^{1/2}\rho v^2 L^2,$$

where C is the modified drag coefficient. So

$$v \sim \left(\frac{Mg}{C^{1/2}\rho L^2}\right)^{1/2}.$$

For geometrically similar animals, g is independent of size (they all fight the same gravity) and C is also independent of size (because the drag coefficient depends only on shape). But M depends on L according to $M \propto L^3$ or $L \propto M^{1/3}$. Because L^2 is proportional to $M^{2/3}$, the denominator contains $M^{2/3}$. The numerator contains M^1, so the ratio of numerator to denominator is $M^{1/3}$. After taking the square root, we find the scaling

$$v \propto M^{1/6}.$$

In other words, $\beta = 1/6$.

Large birds (and planes) fly slightly faster than small birds and planes. The design of the 737 was affected by this fact. The 737 is for medium-range flights and carries fewer passengers than a 747. However, if the 737 were merely a geometrically scaled 747 – retaining the shape but reducing M by, say, a factor of 3 – then it would have a cruising speed roughly 20% lower than a 747 (because $3^{1/6} \approx 1.2$). That reduction would be fine if the 737 were the only plane traveling the skies. But planes are directed along fixed flight paths where it is dangerous to have planes overtaking one another. Therefore, the 737 was designed not to be geometrically similar to the 747 but instead to have the same cruising speed as the 747. Scaling matters!

Problem 10 Speed of a bar-tailed godwit

Use the results of **Problem 8** and **Problem 9** to write the ratio $v_{747}/v_{\text{godwit}}$ as a product of dimensionless factors, where v_{747} is the minimum-energy speed of a 747, and v_{godwit} is the minimum-energy speed of a bar-tailed godwit (i.e. its cruising speed). By estimating the dimensionless factors and their product, estimate the cruising speed of a bar-tailed godwit. [Useful information: $m_{\text{godwit}} \sim 0.4\,\text{kg}$; $v_{747} \sim 600\,\text{mph}$.]

$10^{\square} \pm \square\ \text{m}\,\text{s}^{-1}$ ***or*** $10^{\square} \ldots \square\ \text{m}\,\text{s}^{-1}$

To include in the explanation box: Compare your result with the speed of the record-setting bar-tailed godwit, which made its 11,570 km journey in 8.5 days.

Assuming that the animals and planes fly at the minimum-energy speed,

$$\frac{v_{747}}{v_{\text{godwit}}} = \left(\frac{\rho_{\text{high}}}{\rho_{\text{sea level}}}\right)^{-1/2} \times \left(\frac{m_{747}}{m_{\text{godwit}}}\right)^{1/6}.$$

A plane flies at around 10 km where the density is roughly one-third of the sea-level density. The mass of a 747 is roughly $4 \cdot 10^5$ kg, so the mass ratio between a 747 and a godwit is 10^6. Therefore, the speed ratio is roughly

$$\frac{v_{747}}{v_{\text{godwit}}} \sim (1/3)^{-1/2} \times (10^6)^{1/6} = \sqrt{3} \times 10 \sim 17.$$

A 747 flies at around 550 mph so the godwit should fly around 550/17 mph ~ 32 mph. The actual speed of record-setting godwit is almost identical:

$$v_{\text{actual}} \sim \frac{11,570\,\text{km}}{8.5\,\text{days}} \times \frac{0.6\,\text{mi}}{1\,\text{km}} \times \frac{1\,\text{day}}{24\,\text{hours}} \sim 35\,\text{mph}.$$