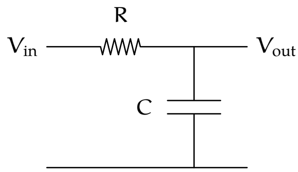
2.3 Low-pass filters

The next example is an analysis that originated in the study of circuits (Section 2.3.1). After those ontological bonds are snipped – once the subject is "considered independently of its original associations" – the core idea (the abstraction) will be useful in understanding diverse natural phenomena including temperature fluctuations (Section 2.3.2).

2.3.1 RC circuits



Linear circuits are composed of resistors, capacitors, and inductors. Resistors are the only time-independent circuit element. To get time-dependent behavior – in other words, to get any interesting behavior – requires inductors or capacitors. Here, as one of the simplest and most widely applicable circuits, we will analyze the behavior of an *RC* circuit.

The input signal is the voltage V_0, a function of time t. The input signal passes through the *RC* system and produces the output signal $V_1(t)$. The differential equation that describes the relation between V_0 and V_1 is (from 8.02)

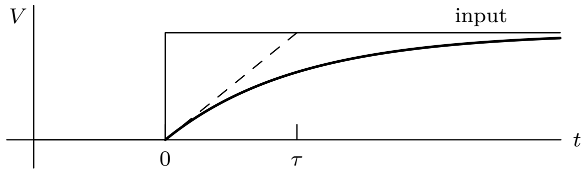
$$\frac{dV_1}{dt} + \frac{V_1}{RC} = \frac{V_0}{RC}. \tag{2.11}$$

This equation contains R and C only as the product RC. Therefore, it doesn't matter what R and C individually are; only their product RC matters. Let's make an abstraction and define a quantity τ as $\tau \equiv RC$.

This time constant has a physical meaning. To see what it is, give the system the simplest nontrivial input: V_0, the input voltage, has been zero since forever; it suddenly becomes a constant V at $t = 0$; and it remains at that value forever ($t > 0$). What is the output voltage V_1? Until $t = 0$, the output is also zero. By inspection, you can check that the solution for $t \geq 0$ is

$$V_1 = V\left(1 - e^{-t/\tau}\right). \tag{2.12}$$

In other words, the output voltage exponentially approaches the input voltage. The rate of approach is determined by the time constant τ. In particular, after one time constant, the gap between the output and input voltages shrinks by a factor of e. Alternatively, if the rate of approach remained its initial value, in one time constant the output would match the input (dotted line).



The actual inputs provided by the world are more complex than a step function. But many interesting real-world inputs are oscillatory (and it turns out that any input can be constructed by adding oscillatory inputs). So let's analyze the effect of an oscillatory input $V_0(t) = Ae^{i\omega t}$, where A is a (possibly complex) constant called the amplitude, and ω is the angular frequency of the oscillations. That complex-exponential notation really means that the voltage is the real part of $Ae^{i\omega t}$, but the 'real part' notation gets distracting if it is repeated in every equation, so traditionally it is omitted.

The RC system is linear – it is described by a linear differential equation – so the output will also oscillate with the same frequency ω. Therefore, write the output in the form $Be^{i\omega t}$, where B is a (possibly complex) constant. Then substitute V_0 and V_1 into the differential equation

$$\frac{dV_1}{dt} + \frac{V_1}{RC} = \frac{V_0}{RC}. \tag{2.13}$$

After removing a common factor of $e^{i\omega t}$, the result is

$$Bi\omega + \frac{B}{\tau} = \frac{A}{\tau}, \tag{2.14}$$

or

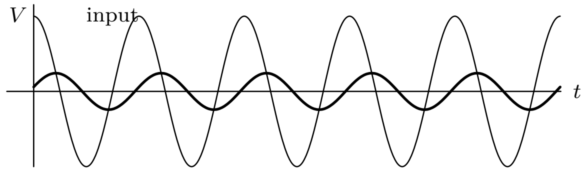
$$B = \frac{A}{1 + i\omega\tau}. \tag{2.15}$$

This equation – a so-called transfer function – contains many generalizable points. First, $\omega\tau$ is a dimensionless quantity. Second, when $\omega\tau$ is small and is therefore negligible compared to the 1 in the denominator, then $B \approx A$. In other words, the output almost exactly tracks the input.

Third, when $\omega\tau$ is large, then the 1 in the denominator is negligible, so

$$B \approx \frac{A}{i\omega\tau}. \tag{2.16}$$

In this limit, the output variation (the amplitude B) is shrunk by a factor of $\omega\tau$ in comparison to the input variation (the amplitude A). Furthermore, because of the i in the denominator, the output oscillations are delayed by 90° relative to the input oscillations (where 360° is a full period). Why 90°? In the complex plane, dividing by i is equivalent to rotating clockwise by 90°. As an example of this delay, if $\omega\tau \gg 1$ and the input voltage oscillates with a period of 4 hr, then the output voltage peaks roughly 1 hr after the input peaks. Here is an example with $\omega\tau = 4$:



In summary, this circuit allows low-frequency inputs to pass through to the output almost unchanged, and it attenuates high-frequency inputs. It is called a low-pass filter: It passes low frequencies and blocks high frequencies. The idea of a low-pass filter, now that we have abstracted it away from its origin in circuit analysis, has many applications.

2.3.2 Temperature fluctuations

The abstraction of a low-pass filter resulting from the solutions to the RC differential equation are transferable. The RC circuit is, it turns out, a model for heat flow; therefore, heat flow, which is everywhere, can be understood by using low-pass filters. As an example, I often prepare a cup of tea but forget to drink it while it is hot. Slowly it cools toward room temperature and therefore becomes undrinkable. If I neglect the cup for still longer – often it spends the night in the microwave, where I forgot it – it warms and cools with the room (for example, it will cool at night as the house cools). A simple model of its heating and cooling is that heat flows in and out through the walls of the mug: the so-called thermal resistance. The heat is stored in the water and mug, which form a heat reservoir: the so-called thermal capacitance. Resistance and capacitance are transferable abstractions.

If R_t is the thermal resistance and C_t is the thermal capacitance, their product R_tC_t is, by analogy with the RC circuit, a thermal time constant τ. To measure it, heat up a mug of tea and watch how the temperature falls toward room temperature. The time for the temperature gap to fall by a factor of e is the time constant τ. In my extensive experience of neglecting cups of tea, in 0.5 hr an enjoyably hot cup of tea becomes lukewarm. To give concrete temperatures to it, 'enjoyably warm' is perhaps 130 °F, room temperature is 70 °F, and lukewarm is perhaps 85 °F. The temperature gap between the tea and the room started at 60 °F and fell to 15 °F – a factor of 4 decrease. It might have required 0.3 hr to have fallen by a factor of e (roughly 2.72). This time is the time constant.

How does the teacup respond to daily temperature variations? In this system, the input signal is the room's temperature; it varies with a frequency of $f = 1\,\text{day}^{-1}$. The output signal is the tea's temperature. The dimensionless parameter $\omega\tau$ is, using $\omega = 2\pi f$, given by

$$\underbrace{2\pi f}_{\omega}\,\tau = 2\pi \times \underbrace{1\,\text{day}^{-1}}_{f} \times \underbrace{0.3\,\text{hr}}_{\tau} \times \frac{1\,\text{day}}{24\,\text{hr}}, \tag{2.17}$$

or approximately 0.1. In other words, the system is driven slowly (ω is not large enough to make $\omega\tau$ near 1), so slowly that the inside temperature almost exactly follows the outside temperature.

A situation showing the opposite extreme of behavior is the response of a house to daily temperature variations. House walls are thicker than teacup walls. Because thermal resistance, like electrical resistance, is proportional to length, the house walls give the house a large thermal resistance. However, the larger surface area of the house compared to the teacup more than compensates for the wall thickness, giving the house a smaller overall thermal resistance. Compared to the teacup, the house has a much, much higher mass and much higher thermal capacitance. The resulting time constant R_tC_t is much longer for the house than for the teacup. One study of houses in Greece quotes 86 hr or roughly 4 days as the thermal time constant. That time constant must be for a well insulated house.

In Cape Town, South Africa, where the weather is mostly warm and houses are often not heated even in the winter, the badly insulated house in which I lived had a thermal time constant of around 0.5 day. The dimensionless parameter $\omega\tau$ is then

$$\underbrace{2\pi f}_{\omega}\,\tau = 2\pi \times \underbrace{1\,\text{day}^{-1}}_{f} \times \underbrace{0.5\,\text{day}}_{\tau}, \qquad (2.18)$$

or approximately 3. In the (South African) winter, the outside temperature varied between 45 °F and 75 °F. This 30 °F outside variation gets shrunk by a factor of 3, giving an inside variation of 10 °F. This variation occurred around the average outside temperature of 60 °F, so the inside temperature varied between 55 °F and 65 °F. Furthermore, if the coldest outside temperature is at midnight, the coldest inside temperature is delayed by almost 6 hr (the one-quarter-period delay). Indeed, the house did feel coldest early in the morning, just as I was getting up – as predicted by this simple model of heat flow that is based on a circuit-analysis abstraction.