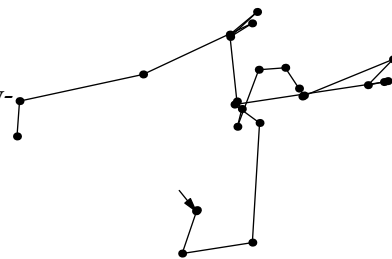


## 8.3 Random walks

Random walks are everywhere. Do you remember the card game War? How long does it last, on average? A molecule of neurotransmitter is released from a vesicle. Eventually it binds to the synapse; then your leg twitches. How long does the molecule take to arrive? On a winter day, you stand outside wearing only a thin layer of clothing. Why do you feel cold?

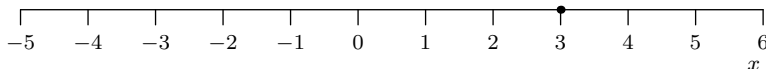
These physical situations are examples of random walks – for example, a gas molecule moving and colliding. The analysis in this section is in three parts. First, we figure out how random walks behave. Then we use that knowledge to derive the diffusion equation, which is a reusable idea (an abstraction). Finally, we apply the diffusion equation to heat flows (keeping warm on a cold day).



### 8.3.1 Behavior of regular walks

In a general random walk, the walker can move a variable distance and in any direction. This general situation is complicated. Fortunately, the essential features of the random walk do not depend on these complicated details. Let's simplify. The complexity arises from the generality – namely, because the direction and the distance between collisions are continuous. To simplify, lump the possible distances: Assume that the particle can travel only a fixed distance between collisions. In addition, lump the possible directions: Assume that the particle can travel only along coordinate axes. Further specialize by analyzing the special case of one-dimensional motion before going to the more general cases of two- and three-dimensional motion.

In this lumped one-dimensional model, a particle starts at the origin and moves along a line. At each tick it moves left or right with probability  $1/2$  in each direction. Here it is at  $x = 3$ :



Let the position after  $n$  steps be  $x_n$ , and the expected position after  $n$  steps be  $\langle x_n \rangle$ . The expected position is the average of all its possible positions, weighted by their probabilities. Because the random walk is unbiased – motion in each direction is equally likely – the expected position cannot change (that’s a symmetry argument).

$$\langle x_n \rangle = \langle x_{n-1} \rangle.$$

Therefore,  $\langle x \rangle$ , the first moment of the position, is an invariant. Alas, it is not a fascinating invariant because it does not tell us anything that we did not already understand.

Let’s try the next-most-complicated moment: the second moment  $\langle x^2 \rangle$ . Its analysis is easiest in special cases. Suppose that, after wandering a while, the particle has arrived at 7, i.e.  $x = 7$ . At the next tick it will be at either  $x = 6$  or  $x = 8$ . Its expected squared position – *not* its squared expected position! – has become

$$\langle x^2 \rangle = \frac{1}{2} (6^2 + 8^2) = 50.$$

The expected squared position increased by 1.

Let’s check this pattern with a second example. Suppose that the particle is at  $x = 10$ , so  $\langle x^2 \rangle = 100$ . After one tick, the new expected squared position is

$$\langle x^2 \rangle = \frac{1}{2} (9^2 + 11^2) = 101.$$

Yet again  $\langle x^2 \rangle$  has increased by 1! Based on those two examples, the conclusion is that

$$\langle x_{n+1}^2 \rangle = \langle x_n^2 \rangle + 1.$$

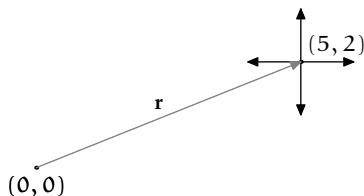
In other words,

$$\langle x_n^2 \rangle = n.$$

Because each step takes the same time (the particle moves at constant speed),

$$\langle x_n^2 \rangle \propto t.$$

The result that  $\langle x^2 \rangle$  is proportional to time applies not only to the one-dimensional random walk. Here's an example in two dimensions. Suppose that the particle's position is  $(5, 2)$ , so  $\langle x^2 \rangle = 29$ . After one step, it has four equally likely positions:



Rather than compute the new expected squared distance using all four positions, be lazy and just look at the two horizontal motions. The two possibilities are  $(6, 2)$  and  $(4, 2)$ . The expected squared distance is

$$\langle x^2 \rangle = \frac{1}{2}(40 + 20) = 30,$$

which is one more than the previous value of  $\langle x^2 \rangle$ . Since nothing is special about horizontal motion compared to vertical motion – symmetry! – the same result holds for vertical motion. So, averaging over the four possible locations produces an expected squared distance of 30.

For two dimensions, the pattern is:

$$\langle x_{n+1}^2 \rangle = \langle x_n^2 \rangle + 1.$$

No step in the analysis depended on being in only two dimensions. In fancy words, the derivation and the result are invariant to change of dimensionality. In plain English, this result also works in three dimensions.

In a standard walk in a straight line,  $\langle x \rangle \propto \text{time}$ . Note the single power of  $x$ . The interesting quantity in a regular walk is not  $x$  itself, since it can grow without limit and is not invariant, but the ratio  $x/t$ , which is invariant to changes in  $t$ . This invariant is known as the speed.

In a random walk, where  $\langle x^2 \rangle \propto t$ , the interesting quantity is  $\langle x^2 \rangle/t$ . The expected squared position is not invariant to changes in  $t$ . However, the ratio  $\langle x^2 \rangle/t$  is invariant. This invariant is, except for a dimensionless constant, the *diffusion constant* and is often denoted  $D$ . It has dimensions of  $L^2T^{-1}$ .

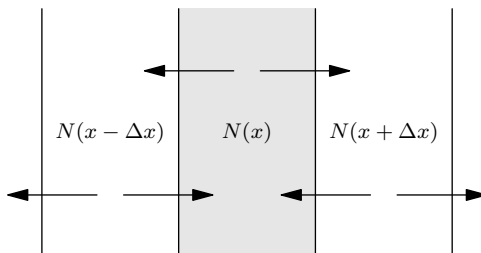
This qualitative difference between a random and a regular walk makes intuitive sense. A random walker, for example a gas molecule or a very

drunk person, moves back and forth, sometimes making progress in one direction, and other times undoing that progress. So, in order to travel the same distance, a random walker should require longer than a regular walker requires. The relation  $\langle x^2 \rangle / t \sim D$  confirms and sharpens this intuition. The time for a random walker to travel a distance  $l$  is  $t \sim l^2 / D$ , which grows quadratically rather than linearly with distance.

### 8.3.2 Diffusion equation

The preceding conclusion about random walks is sufficient to derive the diffusion equation, which describes how charge (electrons) move in a wire, how heat conducts through solid objects, and how gas molecules travel. Imagine then a gas of particles with each particle doing a random walk in one dimension. What is the equation that describes how the concentration, or number density, varies with time?

Divide the one-dimensional world into slices of width  $\Delta x$ , where  $\Delta x$  is the mean free path. Then look at the slices at  $x - \Delta x$ ,  $x$ , and  $x + \Delta x$ . In every time step, one-half the molecules in each slice move left, and one-half move right. So the number of molecules in the  $x$  slice changes from  $N(x)$  to



$$\frac{1}{2}(N(x - \Delta x) + N(x + \Delta x)).$$

The change in  $N$  is

$$\begin{aligned} \Delta N &= \frac{1}{2}(N(x - \Delta x) + N(x + \Delta x)) - N(x) \\ &= \frac{1}{2}(N(x - \Delta x) - 2N(x) + N(x + \Delta x)). \end{aligned}$$

This last relation can be rewritten as

$$\Delta N \sim (N(x + \Delta x) - N(x)) - (N(x) - N(x + \Delta x)).$$

In terms of derivatives, it is

$$\Delta N \sim (\Delta x)^2 \frac{\partial^2 N}{\partial x^2}.$$

The slices are separated by a distance such that most of the molecules travel from one piece to the neighboring piece in the time step  $\tau$ . If  $\tau$  is the time between collisions – the mean free time – then the distance is the mean free path  $\lambda$ . Thus

$$\frac{\Delta N}{\tau} \sim \frac{\lambda^2}{\tau} \frac{\partial^2 N}{\partial x^2},$$

or

$$\dot{N} \sim D \frac{\partial^2 N}{\partial x^2}$$

where  $D \sim \lambda^2/\tau$  is a diffusion constant.

This partial-differential equation has interesting properties. The second spatial derivative means that a linear spatial concentration gradient remains unchanged. Its second derivative is zero so its time derivative must be zero. Diffusion fights curvature – roughly speaking, the second derivative – and does not try to change the gradient directly.

### 8.3.3 Keeping warm

One consequence of the diffusion equation is an analysis of how to keep warm on a cold day. To quantify keeping warm, or feeling cold, we need to calculate the heat flux: the energy flowing per unit area per unit time. Start with the definition of flux. Flux (of anything) is defined as

$$\text{flux of stuff} = \frac{\text{stuff}}{\text{area} \times \text{time}}.$$

The flux depends on the density of stuff and on how fast the stuff travels:

$$\text{flux of stuff} = \frac{\text{stuff}}{\text{volume}} \times \text{speed}.$$

For heat flux, the stuff is thermal energy. The specific heat  $c_p$  is the thermal energy per mass per temperature,  $c_p T$  is the thermal energy per mass, and  $\rho c_p T$  is therefore the thermal energy per volume. The speed is the ‘speed’ of diffusion. To diffuse a distance  $l$  takes time  $t \sim l^2/D$ , making the speed  $l/t$  or  $D/l$ . The  $l$  in the denominator indicates that, as expected, diffusion is slow over long distances. For heat diffusion, the diffusion constant is denoted  $\kappa$  and called the thermal diffusivity. So the speed is  $l/\kappa$ .

Combine the thermal energy per volume with the diffusion speed:

$$\text{thermal flux} = \rho c_p T \times \frac{\kappa}{l}.$$

The product  $\rho c_p \kappa$  occurs so frequently that it is given a name: the thermal conductivity  $K$ . The ratio  $T/l$  is a lumped version of the temperature gradient  $\Delta T/\Delta x$ . With those substitutions, the thermal flux is

$$F = K \frac{\Delta T}{\Delta x}.$$

With one side held at  $T_1$  and the other at  $T_2$ , the temperature gradient is  $(T_2 - T_1)/\Delta x$ .

To estimate how much heat one loses on a cold day, we need to estimate  $K = \rho c_p \kappa$ . To do so, put all the pieces together:

$$\begin{aligned} \rho &\sim 1 \text{ kg m}^{-3}, \\ c_p &\sim 10^3 \text{ J kg}^{-1} \text{ K}^{-1}, \\ \kappa &\sim 1.5 \cdot 10^{-5} \text{ m}^2 \text{ s}^{-1}, \end{aligned}$$

where we are guessing that  $\kappa = \nu$  (because both  $\kappa$  and  $\nu$  are diffusion constants). Then

$$K = \rho c_p \kappa \sim 0.02 \text{ W m}^{-1} \text{ K}^{-1}.$$

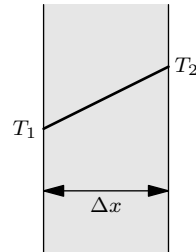
Using this value we can estimate the heat loss on a cold day. Let's say that your skin is at  $T_2 = 30^\circ\text{C}$  and the air outside is  $T_1 = 0^\circ\text{C}$ , making  $\Delta T = 30 \text{ K}$ . A thin T-shirt may have thickness  $2 \text{ mm}$ , so

$$F = K \frac{\Delta T}{\Delta x} \sim 0.02 \text{ W m}^{-1} \text{ K}^{-1} \times \frac{30 \text{ K}}{2 \cdot 10^{-3} \text{ m}} \sim 300 \text{ W m}^{-2}.$$

Damn, we want a power rather than a power per area. Ah, flux is power per area, so just multiply by a person's surface area: roughly  $2 \text{ m}$  tall and  $0.5 \text{ m}$  wide, with a front and a back. So the surface area is about  $2 \text{ m}^2$ . Thus, the power lost is

$$P \sim FA = 300 \text{ W m}^{-2} \times 2 \text{ m}^2 = 600 \text{ W}.$$

No wonder a winter day wearing only thin pants and shirt feels so cold:  $600 \text{ W}$  is large compared to human power levels. Sitting around, a person produces  $100 \text{ W}$  of heat (the basal metabolic rate). When  $600 \text{ W}$  escapes,



one is losing far more than the basal metabolic rate. Eventually, one's core body temperature falls. Then chemical reactions slow down. This happens for two reasons. First, almost all reactions go slower at lower temperature. Second, enzymes lose their optimized shape, so they become less efficient. Eventually you die.

One solution is jogging to generate extra heat. That solution indicates that the estimate of 600 W is plausible. Cycling hard, which generates hundreds of watts of heat, is vigorous enough exercise to keep one warm, even on a winter day in thin clothing.

Another simple solution, as parents repeat to their children: Dress warmly by putting on thick layers. Let's recalculate the power loss if you put on a fleece that is 2 cm thick. You could redo the whole calculation from scratch, but it is simpler is to notice that the thickness has gone up by a factor of 10 but nothing else changed. Because  $F \propto 1/\Delta x$ , the flux and the power drop by a factor of 10. So, wearing the fleece makes

$$P \sim 60 \text{ W.}$$

That heat loss is smaller than the basal metabolic rate, which indicates that one would not feel too cold. Indeed, when wearing a thick fleece, only the exposed areas (hands and face) feel cold. Those regions are exposed to the air, and are protected by only a thin layer of still air (the boundary layer). Because a large  $\Delta x$  means a small heat flux, the moral is (speaking as a parent): Listen to your parents and bundle up!