6.061 Supplementary Notes 2
AC Power Flow In Linear Networks

J.L. Kirtley Jr.

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1 Introduction

This note deals with alternating voltages and currents and with associated energy flows. The focus is on sinusoidal steady state conditions, in which virtually all quantities of interest may be represented by single, complex numbers.

Accordingly, this section opens with a review of complex numbers and with representation of voltage and current as complex amplitudes with complex exponential time dependence. The discussion proceeds, through impedance, to describe a pictorial representation of complex amplitudes, called *phasors*. Power is then defined and, in sinusoidal steady state, reduced to complex form. Finally, flow of power through impedances and a conservation law are discussed.

2 Complex Exponential Notation

Start by recognizing a geometric interpretation for a complex number. If we plot the real part on the horizontal (x) axis and the imaginary part on the vertical (y) axis, then the complex number $\underline{z} = x + jy$ (where $j = \sqrt{-1}$) represents a vector as shown in Figure 1. Note that this vector may be represented not only by its real and imaginary components, but also by a magnitude and a *phase angle*:

$$|\underline{z}| = \sqrt{x^2 + y^2} \tag{1}$$

$$\phi = \arctan(\frac{y}{x}) \tag{2}$$

The basis for complex exponential notation is the celebrated *Euler Relation*:

$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \tag{3}$$

which has a representation as shown in Figure 2.

Now, a comparison of Figures 1 and 2 makes it clear that, with definitions (1) and (2),

$$\underline{z} = x + jy = |\underline{z}|e^{j\phi} \tag{4}$$

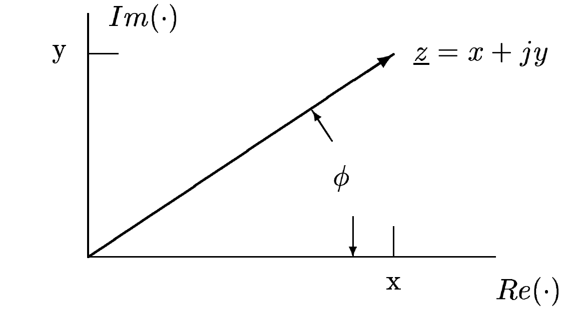

Figure 1: Representation of the complex number $\underline{z} = x + jy$

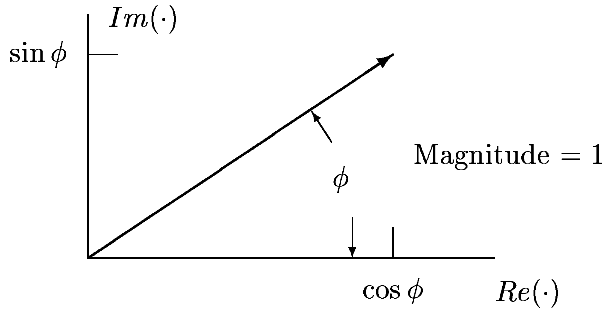

Figure 2: Representation of $e^{j\phi}$

It is straightforward, using (3) to show that:

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2} \tag{5}$$

$$\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j} \tag{6}$$

(7)

The complex exponential is a tremendously useful type of function. Note that the *product* of two numbers expressed as exponenentials is the same as the exponential of the *sums* of the two exponents:

$$e^a e^b = e^{a+b} \tag{8}$$

Note that it is also true that the *reciprocal* of a number in exponential notation is just the exponential of the *negative* of the exponent:

$$\frac{1}{e^a} = e^{-a} \tag{9}$$

Then, if we have two numbers $\underline{z}_1 = |\underline{z}_1|e^{j\phi_1}$ and $\underline{z}_2 = |\underline{z}_2|e^{j\phi_2}$, then the *product* of the two numbers is:

$$\underline{z}_1\underline{z}_2 = |\underline{z}_1||\underline{z}_2|e^{j(\phi_1+\phi_2)} \tag{10}$$

and the *ratio* of the two numbers is:

$$\frac{\underline{z}_1}{\underline{z}_2} = \frac{|\underline{z}_1|}{|\underline{z}_2|}e^{j(\phi_1-\phi_2)} \tag{11}$$

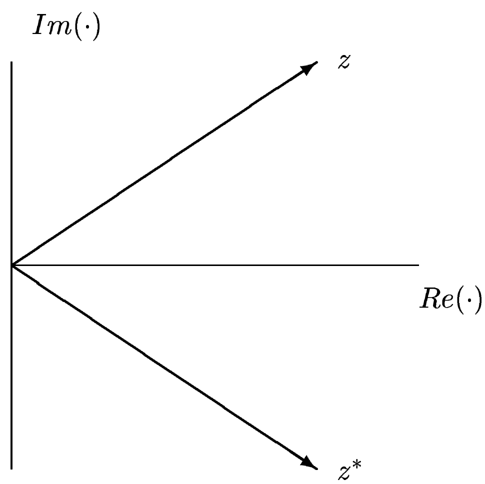

Figure 3: Representation Of A Complex Number And Its Conjugate

The complex conjugate of a number $\underline{z} = x + jy$ is given by:

$$z^* = x - jy \tag{12}$$

The *sum* of a complex number and its conjugate is real:

$$\underline{z} + \underline{z}^* = 2Re(\underline{z}) = 2x \tag{13}$$

while the *difference* is imaginary:

$$\underline{z} - \underline{z}^* = 2jIm(\underline{z}) = 2jy \tag{14}$$

where we have used the two symbols $Re(\cdot)$ and $Im(\cdot)$ to represent the operators which extract the *real* and *imaginary* parts of the complex number.

The complex conjugate of a complex number $\underline{z} = |\underline{z}|e^{j\phi}$ may *also* be written as:

$$\underline{z}^* = |\underline{z}|e^{-j\phi} \tag{15}$$

so that the *product* of a complex number and its conjugate is *real*:

$$\underline{z}\underline{z}^* = |\underline{z}|e^{j\phi}|\underline{z}|e^{-j\phi} = |\underline{z}|^2 \tag{16}$$

3 Sinusoidal Time Functions

A sinusoidal function of time might be written in at least two ways:

$$f(t) = A\cos(\omega t + \phi) \tag{17}$$
$$f(t) = B\cos(\omega t) + C\sin(\omega t) \tag{18}$$

A third way of writing this time function is as the sum of two complex exponentials:

$$f(t) = \underline{X}e^{j\omega t} + \underline{X}^* e^{-j\omega t} \tag{19}$$

Note that the *form* of equation 19, in which complex conjugates are added together, guarantees that the resulting function is *real.*

Now, to relate equation 19 with the other forms of the sinusoidal function, equations 17 and 18, see that $\underline{X}$ may be expressed as:

$$\underline{X} = |\underline{X}|e^{j\psi} \tag{20}$$

Then equation 19 becomes:

$$\begin{aligned} f(t) &= |\underline{X}|e^{j\psi}e^{j\omega t} + |\underline{X}|^* e^{-j\psi}e^{-j\omega t} && (21) \\ &= |\underline{X}|e^{j(\psi+\omega t)} + |\underline{X}|^* e^{-j(\psi+\omega t)} && (22) \\ &= 2|\underline{X}|\cos(\omega t + \psi) && (23) \end{aligned}$$

Then, the coefficients in equation 17 are related to those of equation 19 by:

$$\begin{aligned} |\underline{X}| &= \frac{A}{2} && (24) \\ \psi &= \phi && (25) \end{aligned}$$

Alternatively, we could write

$$\underline{X} = x + jy \tag{26}$$

in which the *real* and *imaginary* parts of $\underline{X}$ are:

$$\begin{aligned} x &= |\underline{X}|\cos(\psi) && (27) \\ y &= |\underline{X}|\sin(\psi) && (28) \end{aligned}$$

Then the time function is written:

$$\begin{aligned} f(t) &= x(e^{j\omega t} + e^{-j\omega t}) + jy(e^{j\omega t} - e^{-j\omega t}) && (29) \\ &= 2x\cos(\omega t) - 2y\sin(\omega t) && (30) \end{aligned}$$

Thus:

$$\begin{aligned} A &= 2x && (31) \\ B &= -2y && (32) \\ X &= \frac{A}{2} - j\frac{B}{2} && (33) \end{aligned}$$

It is also possible to write equation 19 in the form:

$$f(t) = Re(2\underline{X}e^{j\omega t}) \tag{34}$$

While both expressions (19 and 34) are equivalent, it is advantageous to use one or the other of them, according to circumstances. The first notation (equation 19) is the full representation of that sinusoidal signal and may be used under any circumstances. It is, however, cumbersome, so that the somewhat more compact version(equation 34) is usually used. Chiefly when nonlinear products such as power are involved, it is necessary to be somewhat cautions in its use, however, as we will see later on.

4 Impedance

Because it is so easy to differentiate a complex exponential time signal, such a way of representing time signals has real advantages in electric circuits with all kinds of linear elements. In Section 1 of these notes, we introduced the linear *resistance* element, in which voltage and current are linearly related. We must now consider two other elements, inductances and capacitances. The *inductance*

Figure 4: Inductance and Capacitance Elements

produces a relationship between voltage and current which is:

$$v_L = L\frac{di_L}{dt} \tag{35}$$

If voltage and current are sinusoidal functions of time:

$$\begin{aligned} v &= \underline{V}e^{j\omega t} + \underline{V}^* e^{-j\omega t} \\ i &= \underline{I}e^{j\omega t} + \underline{I}^* e^{-j\omega t} \end{aligned}$$

Then the relationship between voltage and current is given simply by:

$$\underline{V} = j\omega L \underline{I} \tag{36}$$

This is a particularly simple form, and as can be seen is directly analogous to *resistance*. We can generalize our view of resistance to *complex impedance* (or simply impedance), in which inductances have impedance which is:

$$\underline{Z}_L = j\omega L \tag{37}$$

The *capacitance* element is similarly defined. A capacitance has a voltage-current relationship:

$$i = C\frac{dv_C}{dt} \tag{38}$$

Thus the *impedance* of a capacitance is:

$$\underline{Z}_C = \frac{1}{j\omega C} \tag{39}$$

The extension to resistive network behavior is now obvious. For problems in *sinusoidal steady state*, in which all excitations are sinusoidal, we may use all of the tricks of linear, resistive network analysis. However, we use *complex* impedance in place of resistance.

The inverse of *impedance* is *admittance*:

$$\underline{Y} = \frac{1}{\underline{Z}}$$

Series and parallel combinations of admittances and impedances are, of course, just like those of conductances and resistances. For two elements in series or in parallel:
Series:

$$\underline{Z} = \underline{Z}_1 + \underline{Z}_2 \tag{40}$$

$$\underline{Y} = \frac{\underline{Y}_1 \underline{Y}_2}{\underline{Y}_1 + \underline{Y}_2} \tag{41}$$

Parallel:

$$\underline{Z} = \frac{\underline{Z}_1 \underline{Z}_2}{\underline{Z}_1 + \underline{Z}_2} \tag{42}$$

$$\underline{Y} = \underline{Y}_1 + \underline{Y}_2 \tag{43}$$

4.1 Example

Suppose we are to find the voltage $v(t)$ in the network of Figure 5, in which $i(t) = I\cos(\omega t)$. The

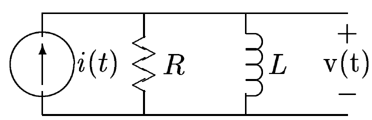

Figure 5: Complex Impedance Network

excitation may be written as:

$$i(t) = \frac{I}{2} e^{j\omega t} + \frac{I}{2} e^{-j\omega t} = Re\left(I e^{j\omega t}\right)$$

Now, the *complex impedance* of the parallel combination of R and L is:

$$R || j\omega L = \frac{R j\omega L}{R + j\omega L}$$

So that, if $v(t)$ is represented by:

$$\begin{aligned} v(t) &= \frac{\underline{V}}{2} e^{j\omega t} + \frac{\underline{V}}{2} e^{-j\omega t} \\ &= Re\left(\underline{V} e^{j\omega t}\right) \end{aligned}$$

Then

$$\underline{V} = \frac{R j\omega L}{R + j\omega L} I$$

Now: the impedance $\underline{Z}$ may be represented by a magnitude and phase angle:

$$
\begin{aligned}
\underline{Z} &= |\underline{Z}|e^{j\phi} \\
|\underline{Z}| &= \frac{\omega LR}{\sqrt{(\omega L)^2 + R^2}} \\
\phi &= \frac{\pi}{2} - \arctan\frac{\omega L}{R}
\end{aligned}
$$

Then, using relations developed here, $v(t)$ may be written as:

$$
v(t) = \frac{\omega LI}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}} \cos(\omega t + \phi)
$$

Note that this expression represents only the *sinusoidal steady state* solution, and therefore does not represent any starting transients.

5 System Functions and Frequency Response

If we are interested in the behavior of a linear system such as the circuits we have been discussing, we often speak of the *system function*. This is the (usually complex) ratio between *output* and *input* of the system. System functions can express *driving point* behavior (impedance or its reciprocal, admittance) or *transfer* behavior. We speak of voltage or current transfer ratios and of transfer impedance (output voltage related to input current) and transfer admittance (output current related to input voltage).

The system function may be expressed in a number of ways, often as a Laplace Transform. Such is beyond the scope of this subject. However, it is important to understand one way of expressing linear system behavior, in the form of *frequency response*. The frequency response of a system is the complex number that relates output of the system to input as a function of frequency. Usually it is expressed as a pair of numbers, magnitude and phase angle. Thus

$$
H(j\omega) = |H(j\omega)|e^{j\phi(j\omega)}
$$

Subjects in Signals and Systems or Network Theory often spend some time on how to obtain and plot the frequency response of a network in ways which are both useful and easy. For our purposes, a straightforward, perhaps even "brute force" approach will do. Consider, for example, the circuit shown in Figure 6.

This is just a voltage divider between an inductance and a resistance. We seek to find, and then plot, the transfer ratio V_{out}/V_{in} of this network. A *very* little analysis yields an expression for the transfer function, which is:

$$
\frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega \frac{L}{R}}
$$

The magnitude and angle of this function can be extracted in a number of ways. For the purpose of these notes, we have done the mathematics using MATLAB. The specific instructions for producing the frequency response plot are shown in Figure 7. Funamentally what is done is to compute the system function for a number of frequencies (note that we use a way of computing specific frequencies which produces a uniform spacing on a logarithmic scale, and then plotting the magnitude (also on a logarithmic scale) and angle of that system function against frequency.

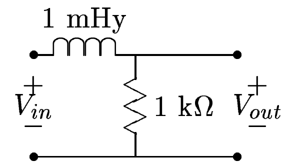

Figure 6: Example Circuit for Frequency Response

6 Phasors

Phasors are *not* weapons. They are a handy geometric trick which help us understand the nature of sinusoidal steady state signals and systems. To start, consider the basis for complex exponential time notation, the function $e^{j\omega t}$. At any instant of time, this is a complex number: at time $t = 0$ it is equal to 1, at time $\omega t = \frac{\pi}{2}$ it is equal to j, and so forth. We may describe this function as a *vector*, of length unity, rotating about the origin of the complex number plane, with angular velocity ω. It has, of course, both *real* and *imaginary* parts, which are just the projections of the vector onto the *real* and *imaginary* axes.

Now consider a sinusoidally varying signal $x(t)$, which may be represented by:

$$x(t) = \frac{\underline{X}}{2} e^{j\omega t} + \frac{\underline{X}^*}{2} e^{-j\omega t}$$

This is the sum of two numbers, complex conjugates, which are, as functions of time, rotating in *opposite directions* in the complex plane. The *sum* of the two is, of course, real. This is the same time function as:

$$x(t) = Re\left(\underline{X} e^{j\omega t}\right) \tag{44}$$

where the real part operator $Re(\cdot)$ simply takes the *projection* of the function on the real axis.

It might be helpful at this point to remember one of the features of complex arithmetic. Multiplication of two complex numbers results in a third complex number which has:

1. a *magnitude* which is the *product* of the magnitudes of the two numbers begin multiplied and,

2. an *angle* which is the *sum* of the angles of the two numbers being multiplied.

Thus, multiplying a number by $e^{j\omega t}$, which has a *magnitude* of unity and an *angle* which is increasing with time at the rate ω, simply has the effect of setting that number spinning around the origin of the complex plane.

It is therefore relatively easy to represent sinusoidally varying signals with just their complex amplitudes, understanding that they also include $e^{j\omega t}$, which provides time variation. The *complex amplitude* includes not only the *magnitude* of the signal, but also a *phase angle*. Usually the phase angle by itself is of little use, and must be related to some time reference. That is, as we will see, it is the *difference* between phase angles that is important in most cases.

Impedances and admittances are also complex numbers, so that *phasors* can be used to visualize the relationship between voltages and currents in a network. The key here is that multiplication and

```
L=1e-3;                              % Set Parameter Values
R=1000;
e=3:.05:7;                           % This is a way of producing evenly
f=10 .^ e;                           % spaced points on a logarithmic chart
om=2*pi .* f;                        % Frequency in radians per second
H = 1 ./ (1 + j*L/R .* om);          % This is the frequency response
subplot(211);
loglog(f, abs(H))                    % Plot of magnitude
xlabel('Frequency, Hz');
ylabel('Magnitude');
grid
subplot(212);
semilogx(f, angle(H))                % Plot of angle
xlabel('Frequency, Hz')
ylabel('Angle')
grid
title('Frequency Response of L-R')
print('freq.ps')
```

Figure 7: MATLAB Program `freq.m`

division of complex numbers is the same as multiplication or division of *magnitudes* and addition or subtraction of *angles*.

6.1 Example

Consider the simple network shown in Figure 9, and suppose that the current source is sinusoidal:

$$i = Re\left(\underline{I}e^{j\omega t}\right)$$

The *impedance* of the R-L combination is a complex number:

$$\underline{Z} = R + j\omega L = 1 + j2$$

Now: the *impedance* may be represented in the complex plane as shown in Figure 10.

Voltage v is given by:

$$v = Re\left(\underline{V}e^{j\omega t}\right)$$

where:

$$\underline{V} = \underline{ZI}$$

Then the relationship between voltage and current is as shown in Figure 11. Note that the phase angle between voltage and current is the same as the phase angle of the impedance.

Note that KVL may be represented graphically in the fashion of Figure 12.

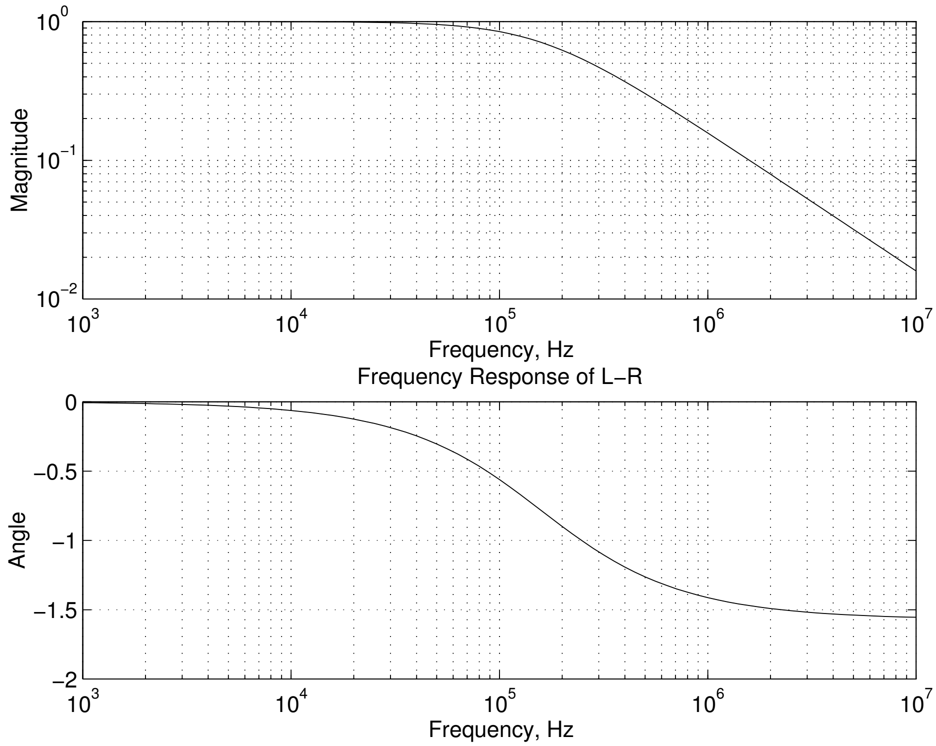

Figure 8: Frequency Response

7 Energy and Power

For any terminal pair with voltage and current defined as shown in Figure 13, *power* flow *into* the element is:

$$p = vi \tag{45}$$

Power is expressed in *Watts* (**W**), and one *Watt* is the product of one *Volt* and one *Ampere.* Energy transferred over an interval of time t_0 to t_1 is the integral of power:

$$w = \int_{t_0}^{t_1} v(t)i(t)dt \tag{46}$$

Energy is expressed in *Joules*, and one *Joule* is one *Watt- Second.* A Joule is also a Newton-Meter (force times distance), and therefore a Watt is a Newton-Meter per Second.

Consider the behavior of the three types of linear, passive elements we have encountered:

- Resistance: $v = Ri$, Instantaneous power is:

$$p = Ri^2 = \frac{v^2}{R} \tag{47}$$

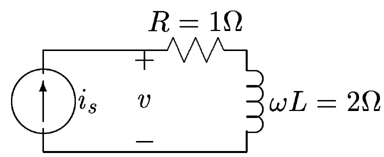

Figure 9: Example Circuit

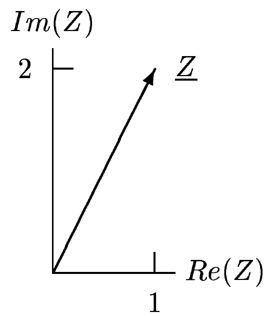

Figure 10: Complex Impedance

- Inductance: $v = L\frac{di}{dt}$, Instantaneous power is:

$$p = iL\frac{di}{dt} = \frac{1}{2}L\frac{di^2}{dt} \tag{48}$$

  The quantity $w_L = \frac{1}{2}Li^2$ may be interpreted as energy stored in the inductance, so that $p = \frac{dw_L}{dt}$. We will need to refine this definition later, when we consider electromechanical interactions or nonlinear elements, but it will do for now.

- Capacitance: $i = C\frac{dv}{dt}$, Instantaneous power is:

$$p = vC\frac{dv}{dt} = \frac{1}{2}C\frac{dv^2}{dt} \tag{49}$$

  The quantity $w_C = \frac{1}{2}Cv^2$ may similarly be interpreted as energy stored in the capacitance.

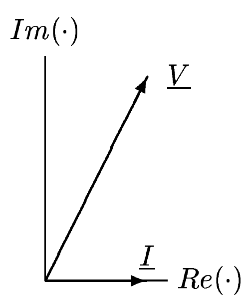

Figure 11: Voltage and Current

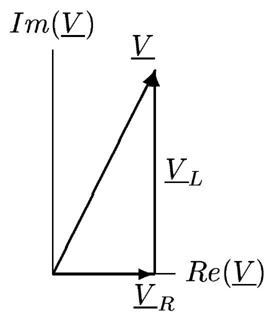

Figure 12: Components of Voltage

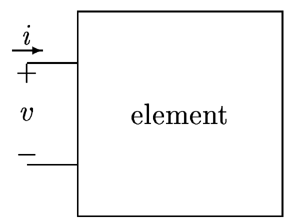

Figure 13: Definition for Power

Next, consider the power input to each of these three elements under sinusoidal steady state conditions:

- Resistance: if $i = I\cos(\omega t + \theta)$, then

$$\begin{aligned} p &= RI^2\cos^2(\omega t + \theta) \\ &= \frac{RI^2}{2}\left[1 + \cos 2(\omega t + \theta)\right] \end{aligned} \tag{50}$$

  Thus, *average* power into the resistance is:

$$P = \frac{1}{2}RI^2 \tag{51}$$

- Inductance: if $i = I\cos(\omega t + \theta)$, then voltage is $v = -\omega LI\sin(\omega t + \theta)$, and power is:

$$\begin{aligned} p &= -\omega LI^2\cos(\omega t + \theta)\sin(\omega t + \theta) \\ &= -\frac{\omega LI^2}{2}\sin 2(\omega t + \theta) \end{aligned} \tag{52}$$

  Average power into the inductance is zero. Instantaneous energy stored in the inductance is

$$w_L = \frac{1}{2}LI^2\cos^2(\omega t + \theta)$$

and *that* has an average value:

$$< w_L >= \frac{1}{4} L I^2 \tag{53}$$

- Capacitance: if $v = V\cos(\omega t + \phi)$, then $i = -\omega C V \sin(\omega t + \phi)$, and power is:

$$p = -\frac{\omega C V^2}{2} \sin 2(\omega t + \phi) \tag{54}$$

which has zero time average. Energy stored in the capacitance is:

$$w_C = \frac{1}{2} C V^2 \cos^2(\omega t + \phi)$$

which has time average:

$$< w_C >= \frac{1}{4} C V^2 \tag{55}$$

Now, consider power flow into a set of terminals in a situation in which both voltage and current are sinusoidal and have the same frequency, but possibly different phase angles:

$$\begin{aligned} v(t) &= V\cos(\omega t + \phi) \\ i(t) &= I \sin(\omega t + \theta) \end{aligned}$$

It is necessary to revert to the original form of complex notation, as in equation 19, to compute power.

$$v(t) = \frac{1}{2}\left[\underline{V} e^{j\omega t} + \underline{V}^* e^{-j\omega t}\right] \tag{56}$$

$$i(t) = \frac{1}{2}\left[\underline{I} e^{j\omega t} + \underline{I}^* e^{-j\omega t}\right] \tag{57}$$

Instantaneous power is the product of voltage and current:

$$p = \frac{1}{4}\left[\underline{V}\,\underline{I}^* + \underline{V}^*\underline{I} + \underline{V}\,\underline{I} e^{j2\omega t} + \underline{V}^*\underline{I}^* e^{-j2\omega t}\right] \tag{58}$$

This is directly equivalent to:

$$p = \frac{1}{2} Re\left[\underline{V}\,\underline{I}^* + \underline{V}\,\underline{I} e^{j2\omega t}\right] \tag{59}$$

This is, in turn, expressible as:

$$p = \frac{1}{2}|\underline{V}||\underline{I}|\left[\cos(\phi - \theta) + \cos(2\omega t + \phi + \theta)\right] \tag{60}$$

From this, we extract "real power", or time- average power:

$$P = \frac{1}{2} Re\left[\underline{V}\,\underline{I}^*\right] = \frac{1}{2}|\underline{V}||\underline{I}|\cos(\phi - \theta) \tag{61}$$

The ratio between *real* power and *apparent power* $P_a = \frac{1}{2}|\underline{V}||\underline{I}|$ is called the *power factor*, and is simply:

$$\text{power factor} = \cos\psi = \cos(\phi - \theta) \tag{62}$$

The *power factor angle* $\psi = \phi - \theta$ is the *relative* phase shift between voltage and current.

This expression for time- average power suggests a definition for something we might call *complex power*:

$$P + jQ = \frac{1}{2}\underline{V}\underline{I}^* \tag{63}$$

in which average power P is the real part. The magnitude of this complex quantity is the *apparent power*. The *imaginary* part is called *reactive power*. It has importance which will be discussed later.

Different *units* are used for real, reactive and apparent power, in order to gain some distinction between quantities. Usually we will express *real power* in *watts* (**W**) (or kW, MW,...). *Apparent power* is expressed in *volt-amperes* (**VA**), and *reactive power* is expressed in *volt-amperes-reactive* (**VAR's**).

To obtain some more feeling for reactive power, expand the time- varying part of the expression for instantaneous power:

$$p_{varying} = \frac{1}{2}|\underline{V}||\underline{I}|\cos(2\omega t + \phi + \theta)$$

Now, using the trig identity $\cos(x + y) = \cos x \cos y - \sin x \sin y$, and assigning $x = 2\omega t + 2\phi$ and $y = -\psi = \theta - \phi$, we have:

$$p_{varying} = \frac{1}{2}|\underline{V}||\underline{I}|\left[\cos 2(\omega t + \phi) + \sin\psi \sin 2(\omega t + \phi)\right]$$

Thus, total instantaneous power is:

$$p = \frac{1}{2}|\underline{V}||\underline{I}|\cos\psi\left[1 + \cos 2(\omega t + \phi)\right] + \frac{1}{2}|\underline{V}||\underline{I}|\sin\psi \sin 2(\omega t + \phi) \tag{64}$$

Now, if we note expressions for P and Q, we can re-write this as:

$$p = P\left[1 + \cos 2(\omega t + \phi)\right] + Q \sin 2(\omega t + \phi) \tag{65}$$

Thus, *real power* P represents not only time average power but also the pulsations that go with time average power. *Reactive power* Q represents energy exchange with zero average value.

7.1 RMS Amplitude

Note that, in all of the expressions for power used so far, a factor of $\frac{1}{2}$ appears. This is, of course, because the *average* value of the product of two sinusoids of the same frequency has a value of half of the products of their *peak* amplitudes multiplied by the cosine of the relative phase angle. It has become common to use a different measure of voltage amplitude, which is called *root-mean-square* or simply RMS. The proper definition for the RMS value of a waveform is somewhat complex, but boils down to that value which, if it were DC, would dissipate the same power in a resistor. It is possible to define RMS for *any* periodic waveform. However, since we will be dealing with sinusoids, the definition is even easier. Clearly, since power dissipated in a resistor is, in terms of *peak* amplitudes:

$$P = \frac{1}{2}\frac{|\underline{V}|^2}{R}$$

then the *RMS amplitude* must be:

$$V_{RMS} = \frac{|\underline{V}|}{\sqrt{2}} \tag{66}$$

Then,

$$P = \frac{V_{RMS}^2}{R}$$

As we will see, RMS amplitudes are the default for most situations: when a circuit is described as "120 Volts AC", the designation virtually always means 120 Volts, RMS. The peak amplitude of this is $|\underline{V}| = \sqrt{2} \cdot 120 \approx 170$ volts. Often you will see sinusoidal waveforms expressed in the form:

$$v = \sqrt{2} V_{RMS} \cos(\omega t)$$

in which V_{RMS} is obviously the RMS amplitude.

7.2 Example

Consider the simple network of Figure 14. We will calculate the *instantaneous* power flow into that network in terms we have been discussing. Assume that the voltage source has RMS amplitude

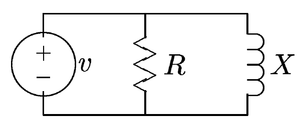

Figure 14: Example Circuit

of 120 volts and R and X are both 100 Ω. Then:

$$v(t) = 170 \cos \omega t$$

The admittance of this network is:

$$Y = \frac{1}{100} - \frac{j}{100}$$

so that the complex amplitude of current is:

$$I = 1.7 - j1.7$$

And then *complex* power is:

$$P + jQ = \frac{1}{2} 170(1.7 + j1.7)$$

Real and reactive power are, respectively: $P = 144$ W, $Q = 144(0,0)VAR$. This gives a *power factor angle* of $\psi = \arctan(1) = 45°$. Then, instantaneous power is:

$$p = 144\left[1 + \cos 2(\omega t - 45°)\right] + 144 \sin 2(\omega t - 45°)$$

This is illustrated in Figure 15.

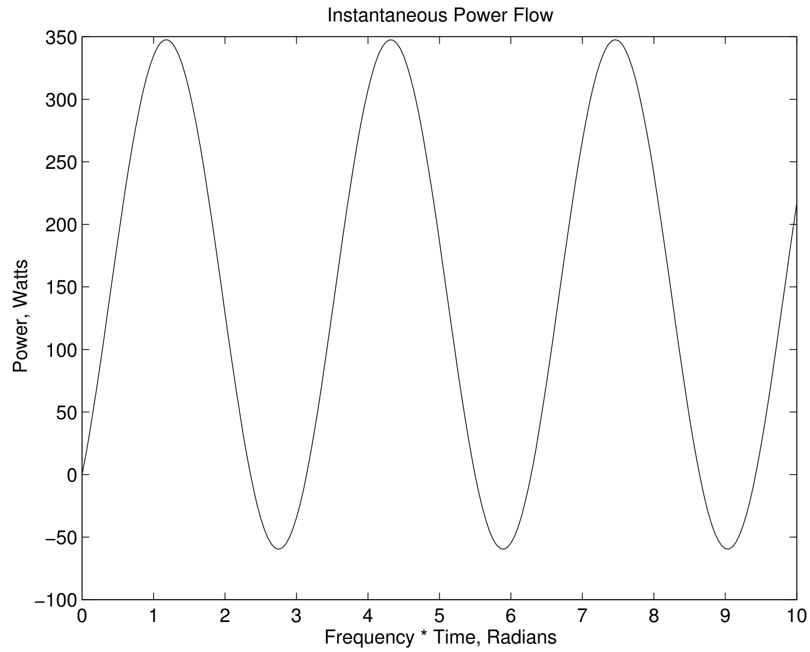

Figure 15: Power Flow For Example Circuit

8 A Conservation Law

It is possible to show that complex power is conserved in the same way as we expect time average power to be conserved. Consider a network with a collection of *terminals* and with a collection of internal *branches*. Instantaneous power flow *into* the network is:

$$p_{in} = \sum_{terminals} vi$$

Note that this expression holds for voltage and current expressed over *any* complete set of terminals. That is, if it is possible to delineate the terminals of the network into a set of *pairs*, the voltages might correspond to voltages across the pair, while currents would flow between the terminals of each pair. Alternatively, the voltages might correspond to single node-to-*datum* voltage, while currents would then be single input node currents. Since power can go *only* into network elements, it follows that the sum of internal branch powers must be equal to the sum of terminal powers:

$$\sum_{terminals} vi = \sum_{branches} vi \tag{67}$$

If this is true for *instantaneous* power, it is also true for *complex* power:

$$\sum_{terminals} \underline{VI} = \sum_{branches} \underline{VI} \tag{68}$$

Now, if the network is made up of resistances, capacitances and inductances,

$$\sum_{terminals} \underline{VI} = \sum_{resistances} \underline{VI} + \sum_{inductances} \underline{VI} + \sum_{capacitances} \underline{VI} \tag{69}$$

For these individual elements:

- Resistances: $\underline{V}\underline{I}^* = R|\underline{I}|^2$
- Inductances: $\underline{V}\underline{I}^* = j\omega L|\underline{I}|^2$
- Capacitances: $\underline{V}\underline{I}^* = -j\omega C|\underline{V}|^2$

Then equation 69 becomes:

$$\sum_{terminals} \underline{V}\underline{I} = \sum_{resistances} R|\underline{I}|^2 + j \sum_{inductances} \omega L|\underline{I}|^2 - j \sum_{capacitances} \omega C|\underline{V}|^2 \tag{70}$$

Then, identifying individual terms:

$$\begin{aligned} \sum_{terminals} \underline{V}\underline{I} &= 2(P + jQ) && \text{Total Complex Power into Network} \\ \sum_{resistances} R|\underline{I}|^2 &= 2\sum < p_r > && \text{Power Dissipated in Resistors} \\ j \sum_{inductances} \omega L|\underline{I}|^2 &= 4\omega \sum < w_L > && \text{Energy Stored in Inductances} \\ j \sum_{capacitances} \omega C|\underline{V}|^2 &= 4\omega \sum < w_C > && \text{Energy Stored in Capacitances} \end{aligned}$$

So, for any RLC network:

$$P + jQ = \sum_{resistors} < p_r > +2j\omega \left[\sum_{inductors} < w_L > - \sum_{capacitors} < w_C > \right] \tag{71}$$

9 Power Flow Through An Impedance

Consider the situation shown in Figure 16. This actually represents a number of important situations in power systems, where the impedance $\underline{Z}$ might represent a transmission line, transformer or motor winding. Of interest to us is the flow of power through the impedance. Current is given

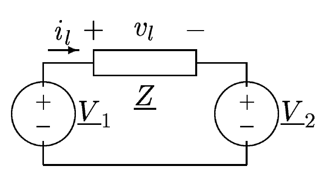

Figure 16: Power Flow Example

by:

$$\underline{i_l} = \frac{\underline{V}_1 - \underline{V}_2}{\underline{Z}} \tag{72}$$

Then, complex power flow out of the left- hand voltage source is:

$$P + jQ = \frac{1}{2}\underline{V}_1 \left(\frac{\underline{V}_1^* - \underline{V}_2^*}{\underline{Z}^*} \right) \tag{73}$$

Now, the complex amplitudes may be expressed as:

$$\underline{V}_1 = |\underline{V}_1|e^{j\theta} \tag{74}$$
$$\underline{V}_2 = |\underline{V}_2|e^{j\theta+\delta} \tag{75}$$

where δ is the *relative* phase angle between the two voltage sources. Complex power at the terminals of the voltage source $\underline{V}_1$ is now given by:

$$P + jQ = \frac{|\underline{V}_1|^2}{2\underline{Z}^*} - \frac{|\underline{V}_1||\underline{V}_2|e^{-j\delta}}{2\underline{Z}^*} \tag{76}$$

This is describable as a circle in the complex plane, with its origin at

$$\frac{|\underline{V}_1|^2}{2\underline{Z}^*}$$

and its radius equal to:

$$\frac{|\underline{V}_1||\underline{V}_2|}{2|\underline{Z}|}$$

A picture of this locus is referred to as a *power circle diagram*, because of its shape.