Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.061 Supplementary Notes 3
Polyphase Networks

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1 Introduction

Most electric power applications employ three phases. That is, three separate power carrying circuite, with voltages and currents staggered symmetrically in time are used. Two major reasons for the use of three phase power are economical use of conductors and nearly constant power flow.

Systems with more than one phase are generally termed *polyphase*. Three phase systems are the most common, but there are situations in which a different number of phases may be used. Two phase systems have a simplicity that makes them useful for teaching vehicles and for certain servomechanisms. This is why two phase machines show up in laboratories and textbooks. Systems with a relatively large number of phases are used for certain specialized applications such as controlled rectifiers for aluminum smelters. Six phase systems have been proposed for very high power transmission applications.

Polyphase systems are qualitatively different from single phase systems. In some sense, polyphase systems are more complex, but often much easier to analyze. This little paradox will become obvious during the discussion of electric machines. It is interesting to note that physical conversion between polyphase systems of different phase number is always possible.

This chapter starts with an elementary discussion of polyphase networks and demonstrates some of their basic features. It ends with a short discussion of per-unit systems and power system representation.

2 Two Phases

The two-phase system is the simplest of all polyphase systems to describe. Consider a pair of voltage sources sitting side by side with:

$$v_1 = V \cos \omega t \quad (1)$$
$$v_2 = V \sin \omega t \quad (2)$$

Suppose this system of sources is connected to al "balanced load", as shown in Figure 1. To compute the power flows in the system, it is convenient to re-write the voltages in complex form:

$$v_1 = Re\left[\underline{V} e^{j\omega t}\right] \quad (3)$$

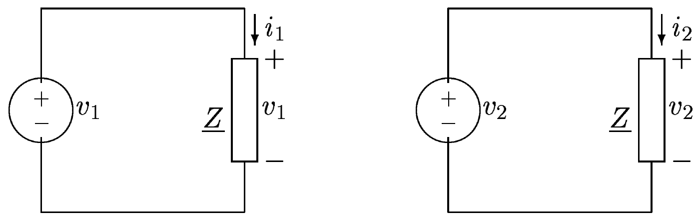

Figure 1: Two-Phase System

$$v_1 = Re\left[-j\underline{V}e^{j\omega t}\right] \tag{4}$$
$$= Re\left[\underline{V}e^{j(\omega t - \frac{\pi}{2})}\right] \tag{5}$$

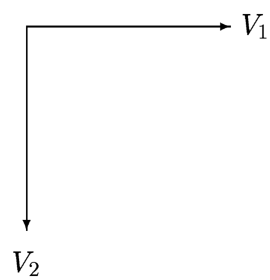

Figure 2: Phasor Diagram for Two-Phase Source

If each source is connected to a load with impedance:

$$\underline{Z} = |\underline{Z}|e^{j\psi}$$

then the complex amplitudes of currents are:

$$\underline{I}_1 = \frac{\underline{V}}{|\underline{Z}|}e^{-j\psi}$$
$$\underline{I}_2 = \frac{\underline{V}}{|\underline{Z}|}e^{-j\psi}e^{-j\frac{\pi}{2}}$$

Each of the two phase networks has the same value for real and reactive power:

$$P + jQ = \frac{|\underline{V}|^2}{2|\underline{Z}|}e^{j\psi} \tag{6}$$

or:

$$P = \frac{|\underline{V}|^2}{2|\underline{Z}|}\cos\psi \tag{7}$$
$$Q = \frac{|\underline{V}|^2}{2|\underline{Z}|}\sin\psi \tag{8}$$

The relationship between "complex power" and instantaneous power flow was worked out in Chapter 2 of these notes. For a system with voltage of the form:

$$v = Re\left[Ve^{j\phi}e^{j\omega t}\right]$$

instantaneous power is given by:

$$p = P\left[1 + \cos 2(\omega t + \phi)\right] + Q \sin 2(\omega t + \phi)$$

For the case under consideration here, $\phi = 0$ for phase 1 and $\phi = -\frac{\pi}{2}$ for phase 2. Thus:

$$\begin{aligned} p_1 &= \frac{|\underline{V}|^2}{2|\underline{Z}|}\cos\psi\left[1 + \cos 2\omega t\right] + \frac{|\underline{V}|^2}{2|\underline{Z}|}\sin\psi \sin 2\omega t \\ p_2 &= \frac{|\underline{V}|^2}{2|\underline{Z}|}\cos\psi\left[1 + \cos(2\omega t - \pi)\right] + \frac{|\underline{V}|^2}{2|\underline{Z}|}\sin\psi \sin(2\omega t - \pi) \end{aligned}$$

Note that the time-varying parts of these two expressions have opposite signs. Added together, they give instantaneous power:

$$p = p_1 + p_2 = \frac{|\underline{V}|^2}{|\underline{Z}|}\cos\psi$$

At least one of the advantages of polyphase power networks is now apparent. The use of a *balanced* polyphase system avoids the power flow pulsations due to ac voltage and current, and even the pulsations due to reactive energy flow. This has obvious benefits when dealing with motors and generators or, in fact, any type of source or load which would like to see constant power.

3 Three Phase Systems

Now consider the arrangement of three voltage sources illustrated in Figure 3.

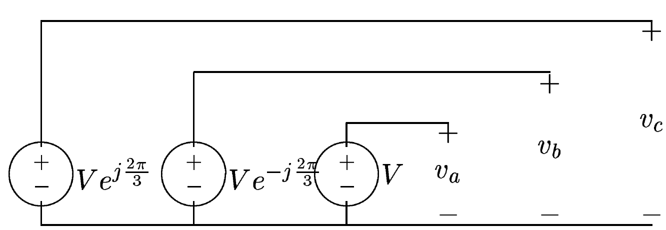

Figure 3: Three- Phase Voltage Source

The three phase voltages are:

$$v_a = V\cos\omega t = Re\left[Ve^{j\omega t}\right] \tag{9}$$

$$v_b = V\cos(\omega t - \tfrac{2\pi}{3}) = Re\left[Ve^{j(\omega t - \frac{2\pi}{3})}\right] \tag{10}$$

$$v_a = V\cos(\omega t + \tfrac{2\pi}{3}) = Re\left[Ve^{j(\omega t + \frac{2\pi}{3})}\right] \tag{11}$$

These three phase voltages are illustrated in the time domain in Figure 4 and as complex phasors in Figure 5. Note the symmetrical spacing in time of the voltages. As in earlier examples, the instantaneous voltages may be visualized by imagining Figure 5 spinning counterclockwise with angular velocity ω. The instantaneous voltages are just projections of the vectors of this "pinwheel" onto the horizontal axis.

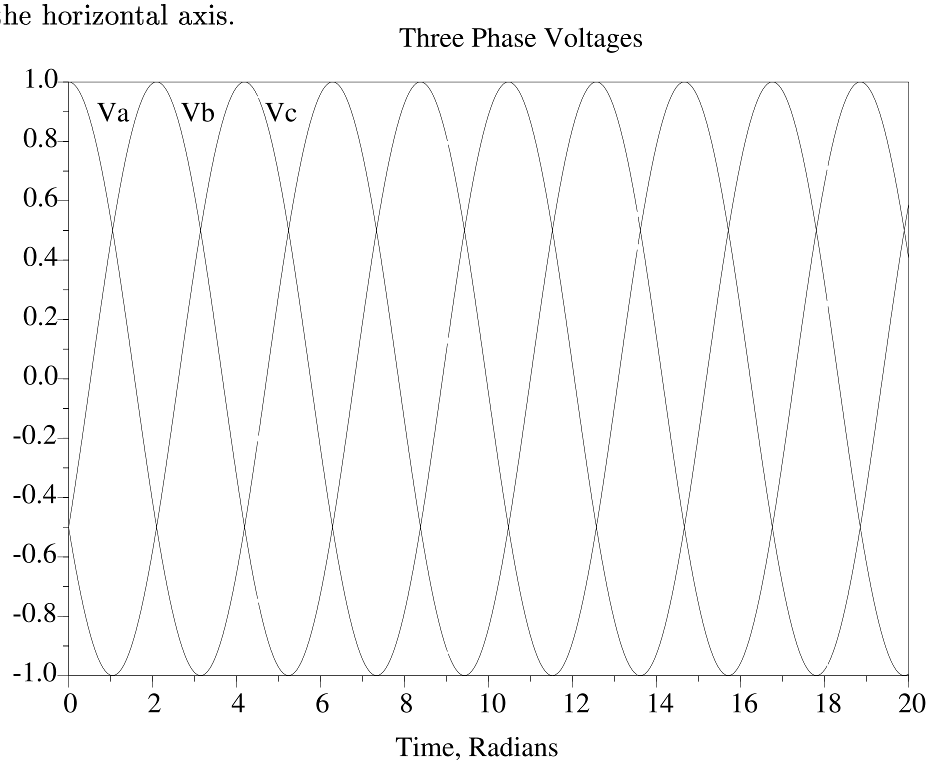

Figure 4: Three Phase Voltages

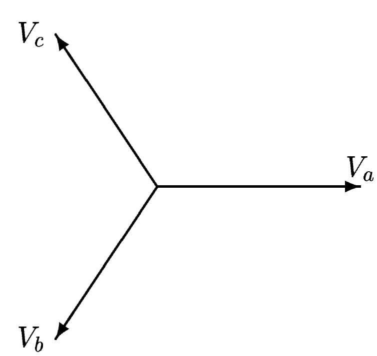

Figure 5: Phasor Diagram: Three Phase Voltages

Consider connecting these three voltage sources to three identical loads, each with complex impedance $\underline{Z}$, as shown in Figure 6.

If voltages are as given by (9 - 11), then currents in the three phases are:

$$i_a = Re\left[\frac{V}{\underline{Z}}e^{j\omega t}\right] \tag{12}$$

$$i_b = Re\left[\frac{V}{\underline{Z}}e^{j(\omega t-\frac{2\pi}{3})}\right] \tag{13}$$

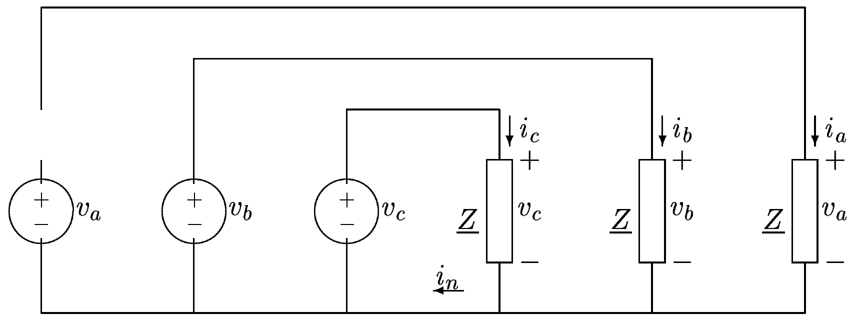

Figure 6: Three- Phase Source Connected To Balanced Load

$$i_c \quad = \quad Re\left[\frac{\underline{V}}{\underline{Z}} e^{j(\omega t + \frac{2\pi}{3})}\right] \tag{14}$$

Complex power in each of the three phases is:

$$P + jQ = \frac{|\underline{V}|^2}{2|\underline{Z}|}\left(\cos\psi + j\sin\psi\right) \tag{15}$$

Then, remembering the time phase of the three sources, it is possible to write the values of instantaneous power in the three phases:

$$p_a \quad = \quad \frac{|\underline{V}|^2}{2|\underline{Z}|}\left\{\cos\psi\left[1 + \cos 2\omega t\right] + \sin\psi \sin 2\omega t\right\} \tag{16}$$

$$p_b \quad = \quad \frac{|\underline{V}|^2}{2|\underline{Z}|}\left\{\cos\psi\left[1 + \cos(2\omega t - \frac{2\pi}{3})\right] + \sin\psi \sin(2\omega t - \frac{2\pi}{3})\right\} \tag{17}$$

$$p_c \quad = \quad \frac{|\underline{V}|^2}{2|\underline{Z}|}\left\{\cos\psi\left[1 + \cos(2\omega t + \frac{2\pi}{3})\right] + \sin\psi \sin(2\omega t + \frac{2\pi}{3})\right\} \tag{18}$$

The sum of these three expressions is total instantaneous power, which is constant:

$$p = p_a + p_b + p_c = \frac{3}{2}\frac{|\underline{V}|^2}{|\underline{Z}|}\cos\psi \tag{19}$$

It is useful, in dealing with three phase systems, to remember that

$$\cos x + \cos(x - \frac{2\pi}{3}) + \cos(x + \frac{2\pi}{3}) = 0$$

regardless of the value of x.

Now consider the current in the neutral wire, i_n in Figure 6. This current is given by:

$$i_n = i_a + i_b + i_c = Re\left[\frac{\underline{V}}{\underline{Z}}\left(e^{j\omega t} + e^{j(\omega t - \frac{2\pi}{3})} + e^{j(\omega t + \frac{2\pi}{3})}\right)\right] = 0 \tag{20}$$

This shows the most important advantage of three-phase systems over two-phase systems: a wire with no current in it does not have to be very large. In fact, the neutral connection may

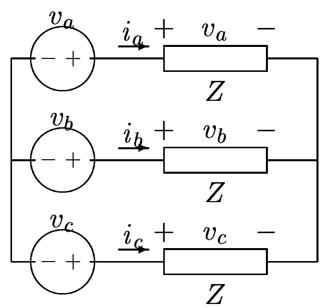

Figure 7: Ungrounded Three-Phase Source and Load

be eliminated completely in many cases. The network shown in Figure 7 will work as well as the network in Figure 6 in most cases in which the voltages and load impedances are balanced.

There is a fundamental difference between grounded and undgrounded systems if perfectly balanced conditions are not maintained. In effect, the ground wire provides isolation between the phases by fixing the neutral voltage a the star point to be zero. If the load impedances are not equal the load is said to be *unbalanced*. If the system is grounded there will be current in the neutral. If an unbalanced load is not grounded, the star point voltage will not be zero, and the voltages will be different in the three phases at the load, even if the voltage sources all have the same magnitude.

4 Line-Line Voltages

A balanced three-phase set of voltages has a well defined set of line-line voltages. If the line-to-neutral voltages are given by (9 - 11), then line-line voltages are:

$$v_{ab} = \quad v_a - v_b = \quad Re\left[\underline{V}\left(1 - e^{-j\frac{2\pi}{3}}\right)e^{j\omega t}\right] \tag{21}$$

$$v_{bc} = \quad v_b - v_c = \quad Re\left[\underline{V}\left(e^{-j\frac{2\pi}{3}} - e^{j\frac{2\pi}{3}}\right)e^{j\omega t}\right] \tag{22}$$

$$v_{ca} = \quad v_c - v_a = \quad Re\left[\underline{V}\left(e^{j\frac{2\pi}{3}} - 1\right)e^{j\omega t}\right] \tag{23}$$

and these reduce to:

$$v_{ab} = Re\left[\sqrt{3}\underline{V}e^{j\frac{\pi}{6}}e^{j\omega t}\right] \tag{24}$$

$$v_{bc} = Re\left[\sqrt{3}\underline{V}e^{-j\frac{\pi}{2}}e^{j\omega t}\right] \tag{25}$$

$$v_{ca} = Re\left[\sqrt{3}\underline{V}e^{j\frac{5\pi}{6}}e^{j\omega t}\right] \tag{26}$$

The phasor relationship of line-to-neutral and line-to-line voltages is shown in Figure 4. Two things should be noted about this relationship:

- The line-to-line voltage set has a magnitude that is larger than the line-ground voltage by a factor of $\sqrt{3}$.
- Line-to-line voltages are phase shifted by 30° ahead of line-to-neutral voltages.

Clearly, line-to-line voltages themselves form a three-phase set just as do line-to-neutral voltages. Power system components (sources, transformer windings, loads, etc.) may be connected either between lines and neutral or between lines. The former connection of often called *wye*, the latter is called *delta*, for obvious reasons.

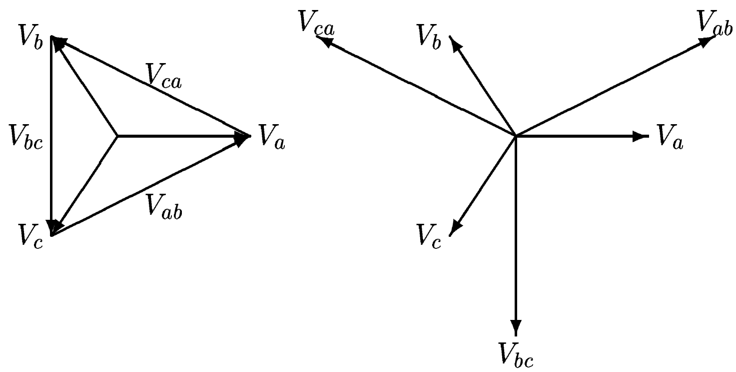

Figure 8: Phasor Diagram: Three Phase Voltages

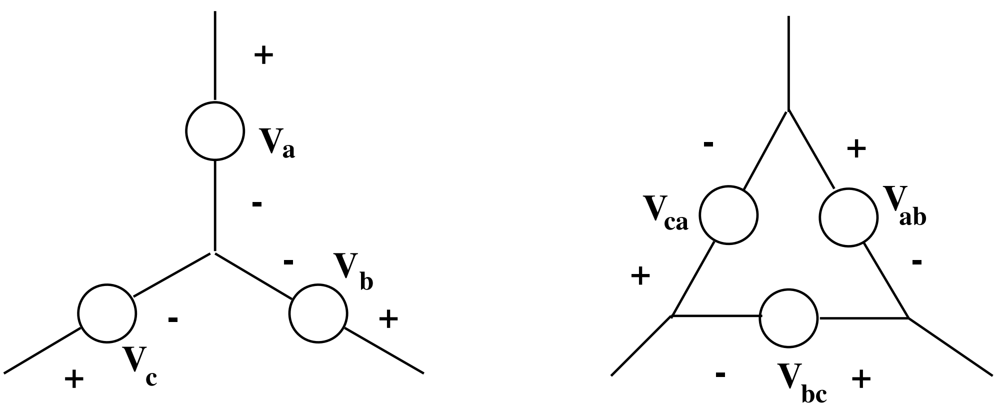

Figure 9: *Wye* And *Delta* Connected Voltage Sources

It should be noted that the *wye* connection is at least potentially a four-terminal connection, while the *delta* connection is inherently three-terminal. The difference is the availability of a neutral point. Under balanced operating conditions this is unimportant, but the difference is apparent and important under unbalanced conditions.

4.1 Example: Wye and Delta Connected Loads

Loads may be connected in either line-to-neutral or line-to-line configuration. An example of the use of this flexibility is in a fairly commonly used distribution system with a line-to-neutral voltage of 120 V, RMS. In this system the line-to-line voltage is 208 V, RMS. Single phase loads may be connected either line-to-line or line-to-neutral.

Suppose it is necessary to build a resistive heater to deliver 6 kW, to be made of three elements which may be connected in either *wye* or *delta*. Each of the three elements must dissipate 2000 W. Thus, since $P = \frac{V^2}{R}$, the *wye* connected resistors would be:

$$R_y = \frac{120^2}{2000} = 7.2\Omega$$

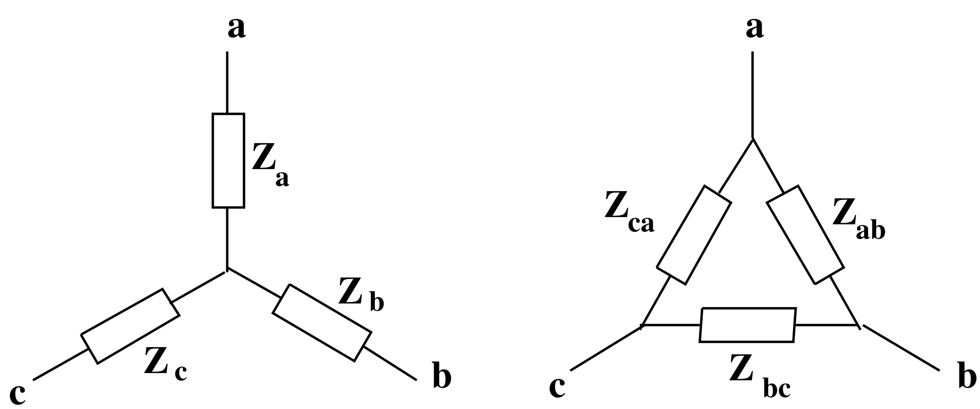

Figure 10: Wye And Delta Connected Impedances

while the *delta* connected resistors would be:

$$R_\Delta = \frac{208^2}{2000} = 21.6\Omega$$

As is suggested by this example, *wye* and *delta* connected impedances are often directly equivalent. In fact, ungrounded connections are three-terminal networks which may be represented in two ways. The two networks shown in Figure 10, combinations of three passive impedances, are directly equivalent and identical in their terminal behavior if the relationships between elements are as given in (27 - 32).

$$\underline{Z}_{ab} = \frac{\underline{Z}_a\underline{Z}_b + \underline{Z}_b\underline{Z}_c + \underline{Z}_c\underline{Z}_a}{\underline{Z}_c} \tag{27}$$

$$\underline{Z}_{bc} = \frac{\underline{Z}_a\underline{Z}_b + \underline{Z}_b\underline{Z}_c + \underline{Z}_c\underline{Z}_a}{\underline{Z}_a} \tag{28}$$

$$\underline{Z}_{ca} = \frac{\underline{Z}_a\underline{Z}_b + \underline{Z}_b\underline{Z}_c + \underline{Z}_c\underline{Z}_a}{\underline{Z}_b} \tag{29}$$

$$\underline{Z}_a = \frac{\underline{Z}_{ab}\underline{Z}_{ca}}{\underline{Z}_{ab} + \underline{Z}_{bc} + \underline{Z}_{ca}} \tag{30}$$

$$\underline{Z}_b = \frac{\underline{Z}_{ab}\underline{Z}_{bc}}{\underline{Z}_{ab} + \underline{Z}_{bc} + \underline{Z}_{ca}} \tag{31}$$

$$\underline{Z}_c = \frac{\underline{Z}_{bc}\underline{Z}_{ca}}{\underline{Z}_{ab} + \underline{Z}_{bc} + \underline{Z}_{ca}} \tag{32}$$

A special case of the *wye-delta* equivalence is that of *balanced* loads, in which:

$$\underline{Z}_a = \underline{Z}_b = \underline{Z}_c = \underline{Z}_y$$

and

$$\underline{Z}_{ab} = \underline{Z}_{bc} = \underline{Z}_{ca} = \underline{Z}_\Delta$$

in which case:

$$\underline{Z}_\Delta = 3\underline{Z}_y$$

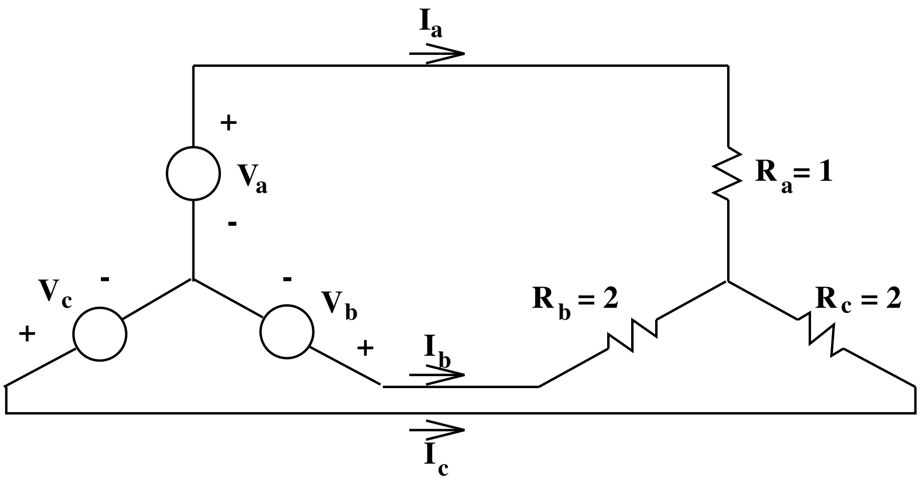

Figure 11: Unbalanced Load

4.2 Example: Use of Wye-Delta for Unbalanced Loads

The unbalanced load shown in Figure 11 is connected to a balanced voltage source. The problem is to determine the line currents. Note that his load is ungrounded (if it *were* grounded, this would be a trivial problem). The voltages are given by:

$$\begin{aligned} v_a &= V \cos \omega t \\ v_b &= V \cos(\omega t - \frac{2\pi}{3}) \\ v_c &= V \cos(\omega t + \frac{2\pi}{3}) \end{aligned}$$

To solve this problem, convert both the source and load to delta equivalent connections, as shown in Figure 12. The values of the three resistors are:

$$r_{ab} = r_{ca} = \frac{2+4+2}{2} = 4$$

$$r_{bc} = \frac{2+4+2}{1} = 8$$

The complex amplitudes of the equivalent voltage sources are:

$$\begin{aligned} \underline{V}_{ab} = &\quad \underline{V}_a - \underline{V}_b = \underline{V}\left(1 - e^{-j\frac{2\pi}{3}}\right) = &\quad \underline{V}\sqrt{3}e^{j\frac{\pi}{6}} \\ \underline{V}_{bc} = &\quad \underline{V}_b - \underline{V}_c = \underline{V}\left(e^{-j\frac{2\pi}{3}} - e^{j\frac{2\pi}{3}}\right) = &\quad \underline{V}\sqrt{3}e^{-j\frac{\pi}{2}} \\ \underline{V}_{ca} = &\quad \underline{V}_c - \underline{V}_a = \underline{V}\left(e^{j\frac{2\pi}{3}} - 1\right) = &\quad \underline{V}\sqrt{3}e^{j\frac{5\pi}{6}} \end{aligned}$$

Currents in each of the equivalent resistors are:

$$\underline{I}_1 = \frac{\underline{V}_{ab}}{r_{ab}} \quad \underline{I}_2 = \frac{\underline{V}_{bc}}{r_{bc}} \quad \underline{I}_3 = \frac{\underline{V}_{ca}}{r_{ca}}$$

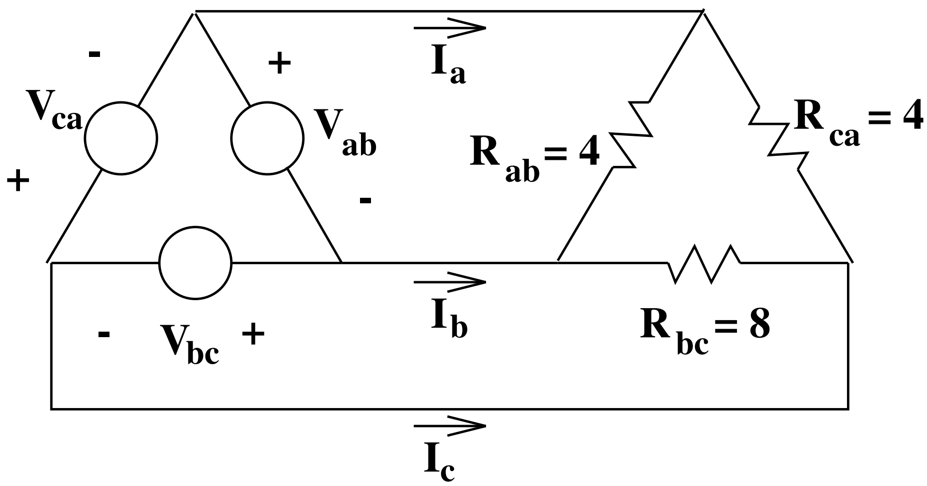

Figure 12: Delta Equivalent

The *line* curents are then just the difference between current in the legs of the delta:

$$
\begin{aligned}
I_a = I_1 - I_3 = &\quad \sqrt{3}V\left(\frac{e^{j\frac{\pi}{6}}}{4} - \frac{e^{j\frac{5\pi}{6}}}{4}\right) &= &\quad \frac{3}{4}V \\
I_b = I_2 - I_1 = &\quad \sqrt{3}V\left(\frac{e^{-j\frac{\pi}{2}}}{8} - \frac{e^{j\frac{\pi}{6}}}{4}\right) &= &\quad -\left(\frac{3}{8} + j\frac{1}{4}\right)V \\
I_c = I_3 - I_2 = &\quad \sqrt{3}V\left(\frac{e^{j\frac{5\pi}{6}}}{4} - \frac{e^{-j\frac{\pi}{2}}}{8}\right) &= &\quad -\left(\frac{3}{8} - j\frac{1}{4}\right)V
\end{aligned}
$$

These are shown in Figure 13.

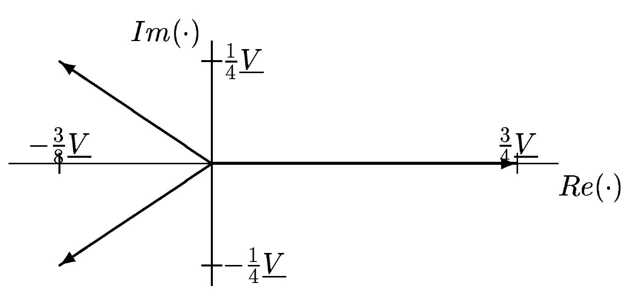

Figure 13: Line Currents

5 Transformers

Transformers are essential parts of most power systems. Their role is to convert electrical energy at one voltage to some other voltage. We will deal with transformers as electromagnetic elements later on in this subject, but for now it will be sufficient to use a simplified model for the transformer which we will call the *ideal* transformer. This is a two-port circuit element, shown in Figure 14.

The *ideal transformer* as a network element constrains its terminal variables in the following way:

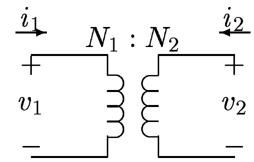

Figure 14: Ideal Transformer

$$\frac{v_1}{N_1} = \frac{v_2}{N_2} \tag{33}$$
$$N_1 i_1 = -N_2 i_2 \tag{34}$$

As it turns out, this is not a terribly bad model for the behavior of a real transformer under most circumstances. Of course, we will be interested in fine points of transformer behavior and behavior under pathological operating conditions, and so will eventually want a better model. For now, it is sufficient to note just a few things about how the transformer works.

1. In normal operation, we select a transformer *turns ratio* $\frac{N_1}{N_2}$ so that the desired voltages appear at the proper terminals. For example, to convert 13.8 kV distribution voltage to the 120/240 volt level suitable for residential or commercial single phase service, we would use a transformer with turns ratio of $\frac{13800}{240} = 57.5$. To split the low voltage in half, a *center tap* on the low voltage winding would be used.

2. The transformer, at least in its *ideal* form, does not consume, produce nor store energy. Note that, according to (33) and (34), the *sum* of power flows into a transformer is identically zero:

$$p_1 + p_2 = v_1 i_1 + v_2 i_2 = 0 \tag{35}$$

3. The transformer also tends to transform impedances. To show how this is, look at Figure 15. Here, some impedance is connected to one side of an ideal transformer. See that it is possible to find an equivalent impedance viewed from the other side of the transformer.

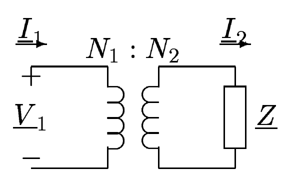

Figure 15: Impedance Transformation

Noting that

$$\underline{I}_2 = -\frac{N_1}{N_2}\underline{I}_1$$

and that

$$\underline{V}_2 = -\underline{Z}\underline{I}_2$$

Then the ratio between input voltage and current is:

$$\underline{V}_1 = \frac{N_1}{N_2}\underline{V}_2 = \left(\frac{N_1}{N_2}\right)^2 \underline{I}_1 \tag{36}$$

6 Three-Phase Transformers

A three-phase transformer is simply three single phase transformers. The complication in these things is that there are a number of ways of winding them, and a number of ways of interconnecting them. We will have more to say about windings later. For now, consider interconnections. On either "side" of a transformer connection (i.e. the *high voltage* and *low voltage* sides), it is possible to connect transformer windings either line to neutral (*wye*), or line to line (*delta*). Thus we may speak of transformer connections being *wye-wye*, *delta-delta*, *wye-delta*, or *delta-wye*.

Ignoring certain complications that we will have more to say about shortly, connection of transformers in either *wye-wye* or *delta-delta* is reasonably easy to understand. Each of the line-to-neutral (in the case of *wye-wye*), or line-to-line (in the case of *delta-delta*) voltages is transformed by one of the three transformers. On the other hand, the interconnections of a *wye-delta* or *delta-wye* transformer are a little more complex. Figure 16 shows a *delta-wye* connection, in what might be called "wiring diagram" form. A more schematic (and more common) form of the same picture is shown in Figure 17. In that picture, winding elements that *appear* parallel are wound on the same core segment, and so constitute a single phase transformer.

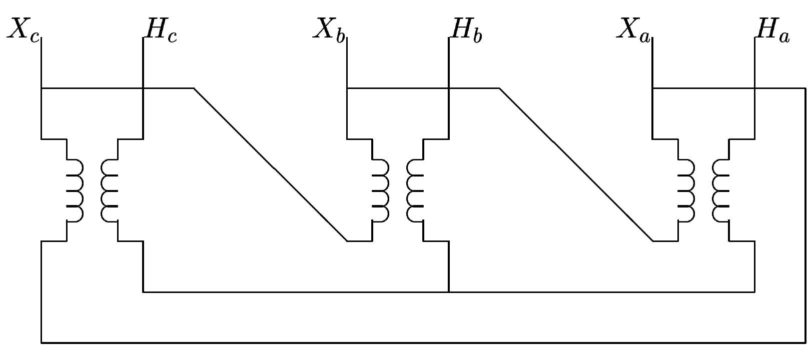

Figure 16: *Delta-Wye* Transformer Connection

Now: assume that N_Δ and N_Y are numbers of turns. If the individual transformers are considered to be ideal, the following voltage and current constraints exist:

$$v_{aY} = \frac{N_Y}{N_\Delta}(v_{a\Delta} - v_{b\Delta}) \tag{37}$$

$$v_{bY} = \frac{N_Y}{N_\Delta}(v_{b\Delta} - v_{c\Delta}) \tag{38}$$

$$v_{cY} = \frac{N_Y}{N_\Delta}(v_{c\Delta} - v_{a\Delta}) \tag{39}$$

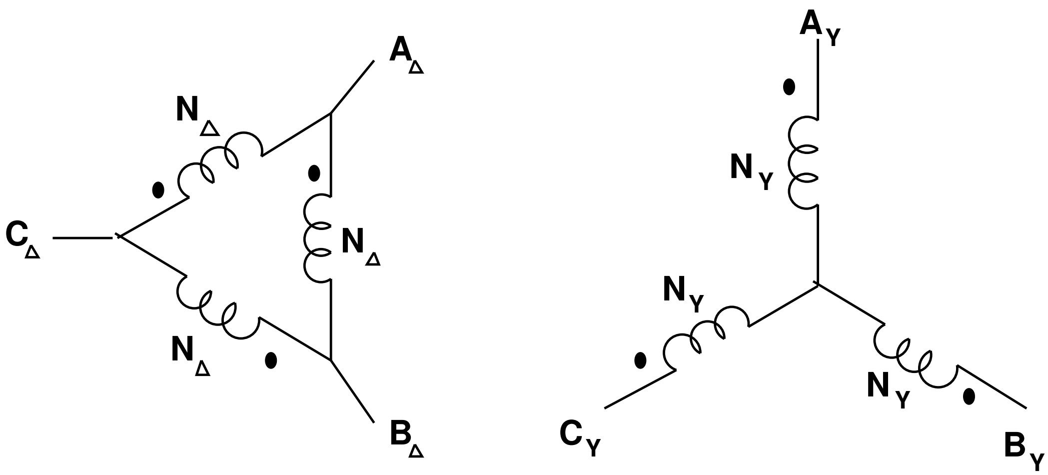

Figure 17: Schematic of *Delta-Wye* Transformer Connection

$$i_{a\Delta} = \frac{N_Y}{N_\Delta}(i_{aY} - i_{cY}) \tag{40}$$

$$i_{b\Delta} = \frac{N_Y}{N_\Delta}(i_{bY} - i_{aY}) \tag{41}$$

$$i_{c\Delta} = \frac{N_Y}{N_\Delta}(i_{cY} - i_{bY}) \tag{42}$$

where each of the *voltages* are line-neutral and the *currents* are in the lines at the transformer terminals.

Now, consider what happens if a $\Delta - Y$ transformer is connected to a balanced three- phase voltage source, so that:

$$\begin{aligned} v_{a\Delta} &= Re\left(\underline{V}e^{j\omega t}\right) \\ v_{b\Delta} &= Re\left(\underline{V}e^{j(\omega t - \frac{2\pi}{3})}\right) \\ v_{c\Delta} &= Re\left(\underline{V}e^{j(\omega t + \frac{2\pi}{3})}\right) \end{aligned}$$

Then, complex amplitudes on the *wye* side are:

$$\begin{aligned} \underline{V}_{aY} &= \tfrac{N_Y}{N_\Delta}\underline{V}\left(1 - e^{-j\frac{2\pi}{3}}\right) = \sqrt{3}\frac{N_Y}{N_\Delta}\underline{V}e^{j\frac{\pi}{6}} \\ \underline{V}_{bY} &= \tfrac{N_Y}{N_\Delta}\underline{V}\left(e^{-j\frac{2\pi}{3}} - e^{j\frac{2\pi}{3}}\right) = \sqrt{3}\frac{N_Y}{N_\Delta}\underline{V}e^{-j\frac{\pi}{2}} \\ \underline{V}_{cY} &= \tfrac{N_Y}{N_\Delta}\underline{V}\left(e^{j\frac{2\pi}{3}} - 1\right) = \sqrt{3}\frac{N_Y}{N_\Delta}\underline{V}e^{j\frac{5\pi}{6}} \end{aligned}$$

Two observations should be made here:

- The ratio of voltages (that is, the ratio of either *line-line* or *line-neutral*) is different from the *turns ratio* by a factor of $\sqrt{3}$.
- All *wye* side voltages are shifted in *phase* by 30° with respect to the *delta* side voltages.

6.1 Example

Suppose we have the following problem to solve:

> A balanced three- phase wye-connected resistor is connected to the Δ side of a $Y-\Delta$ transformer with a nominal *voltage* ratio of
>
> $$\frac{v_\Delta}{v_Y} = N$$
>
> What is the impedance looking into the *wye* side of the transformer, assuming drive with a balanced source?

The situation is shown in Figure 18.

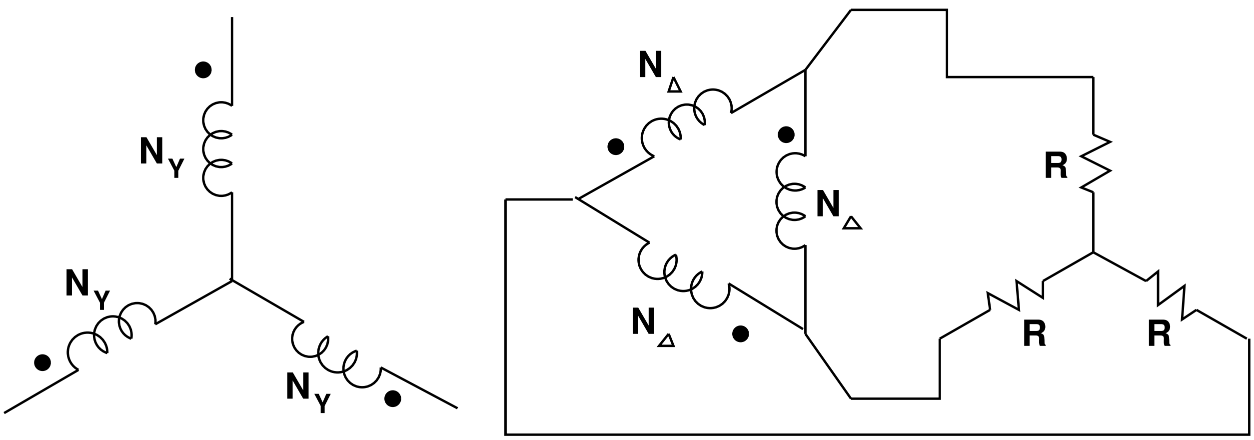

Figure 18: Example

It is important to remember the relationship between the *voltage* ratio and the *turns* ratio, which is:

$$\frac{v_\Delta}{v_Y} = N = \frac{N_\Delta}{\sqrt{3}N_Y}$$

so that:

$$\frac{N_Y}{N_\Delta} = \frac{N}{\sqrt{3}}$$

Next, the $Y-\Delta$ equivalent transform for the load makes the picture look like figure 19

In this situation, each transformer secondary winding is conected directly across one of the three resistors. Currents in the resistors are given by:

$$\begin{aligned} i_1 &= \frac{v_{ab\Delta}}{3R} \\ i_2 &= \frac{v_{bc\Delta}}{3R} \\ i_3 &= \frac{v_{ca\Delta}}{3R} \end{aligned}$$

Line currents are:

$$\begin{aligned} i_{a\Delta} &= i_1 - i_3 = \tfrac{v_{ab\Delta}-v_{ca\Delta}}{3R} &= i_{1\Delta} - i_{3\Delta} \\ i_{b\Delta} &= i_2 - i_1 = \tfrac{v_{bc\Delta}-v_{ab\Delta}}{3R} &= i_{2\Delta} - i_{1\Delta} \\ i_{c\Delta} &= i_3 - i_2 = \tfrac{v_{ca\Delta}-v_{bc\Delta}}{3R} &= i_{3\Delta} - i_{2\Delta} \end{aligned}$$

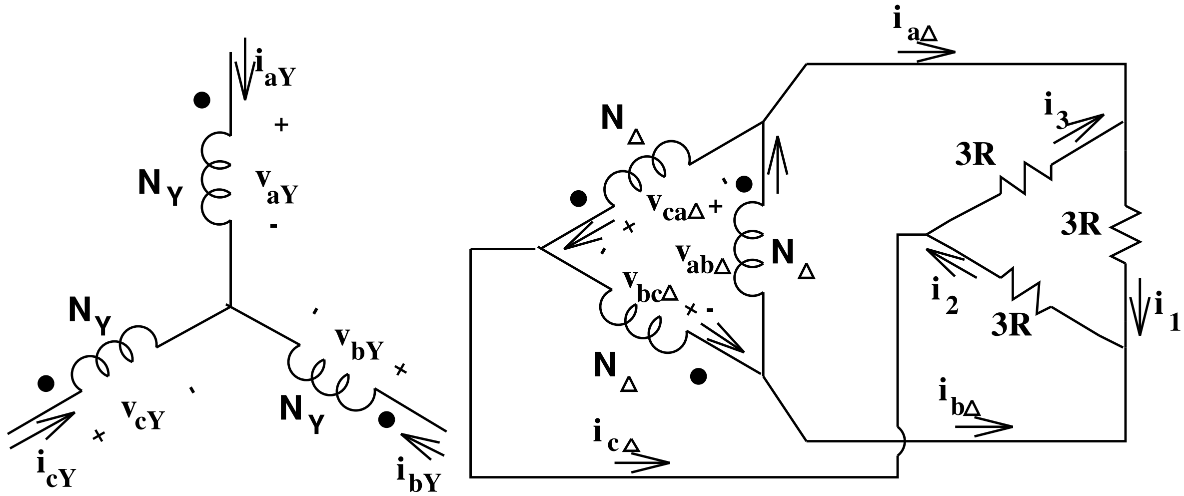

Figure 19: Equivalent Situation

Solving for currents in the legs of the transformer Δ, subtract, for example, the second expression from the first:

$$2i_{1\Delta} - i_{2\Delta} - i_{3\Delta} = \frac{2v_{ab\Delta} - v_{bc\Delta} - v_{ca\Delta}}{3R}$$

Now, taking advantage of the fact that the system is balanced:

$$\begin{aligned} i_{1\Delta} + i_{2\Delta} + i_{3\Delta} &= 0 \\ v_{ab\Delta} + v_{bc\Delta} + v_{ca\Delta} &= 0 \end{aligned}$$

to find:

$$\begin{aligned} i_{1\Delta} &= \frac{v_{ab\Delta}}{3R} \\ i_{2\Delta} &= \frac{v_{bc\Delta}}{3R} \\ i_{3\Delta} &= \frac{v_{ca\Delta}}{3R} \end{aligned}$$

Finally, the ideal transformer relations give:

$$\begin{aligned} v_{ab\Delta} &= \frac{N_\Delta}{N_Y} v_{aY} & \qquad i_{aY} &= \frac{N_\Delta}{N_Y} i_{1\Delta} \\ v_{bc\Delta} &= \frac{N_\Delta}{N_Y} v_{bY} & \qquad i_{bY} &= \frac{N_\Delta}{N_Y} i_{2\Delta} \\ v_{ca\Delta} &= \frac{N_\Delta}{N_Y} v_{cY} & \qquad i_{cY} &= \frac{N_\Delta}{N_Y} i_{3\Delta} \end{aligned}$$

so that:

$$i_{aY} = \left(\frac{N_\Delta}{N_Y}\right)^2 \frac{1}{3R} v_{aY}$$

$$i_{bY} = \left(\frac{N_\Delta}{N_Y}\right)^2 \frac{1}{3R} v_{bY}$$
$$i_{cY} = \left(\frac{N_\Delta}{N_Y}\right)^2 \frac{1}{3R} v_{cY}$$

The apparent resistance (that is, apparent were it to be connected in *wye*) at the *wye* terminals of the transformer is:

$$R_{eq} = 3R\left(\frac{N_Y}{N_\Delta}\right)^2$$

Expressed in terms of *voltage* ratio, this is:

$$R_{eq} = 3R\left(\frac{N}{\sqrt{3}}\right)^2 = R\left(\frac{v_Y}{v_\Delta}\right)^2$$

It is important to note that this solution took the long way around. Taken consistently (uniformly on a line-neutral or uniformly on a line-line basis), impedances transform across transformers by the square of the *voltage* ratio, no matter what connection is used.

7 Polyphase Lines and Single-Phase Equivalents

By now, one might suspect that a balanced polyphase system may be regarded simply as three single-phase systems, even through the three phases are physically interconnected. This feeling is reinforced by the equivalence between *wye* and *delta* connected sources and impedances. One more step is required to show that single phase equivalence is indeed useful, and this concerns situations in which the phases have mutual coupling.

In speaking of *lines*, we mean such system elements as transmission or distribution lines: overhead wires, cables or even in-plant buswork. Such elements have impedance, so that there is some voltage drop between the *sending* and *receiving* ends of the line. This impedance is more than just conductor resistance: the conductors have both *self* and *mutual* inductance, because currents in the conductors make magnetic flux which, in turn, is linked by all conductors of the line.

A schematic view of a *line* is shown in Figure 20. Actually, only the inductance components of line impedance are shown, since they are the most interesting parts of line impedance. Working in

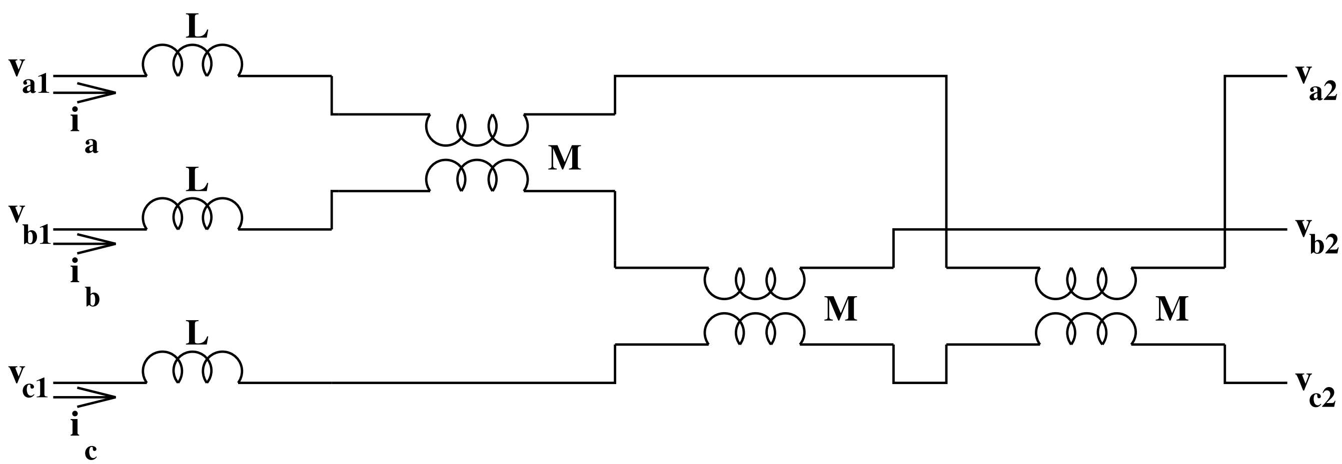

Figure 20: Schematic Of A Balanced Three-Phase Line With Mutual Coupling

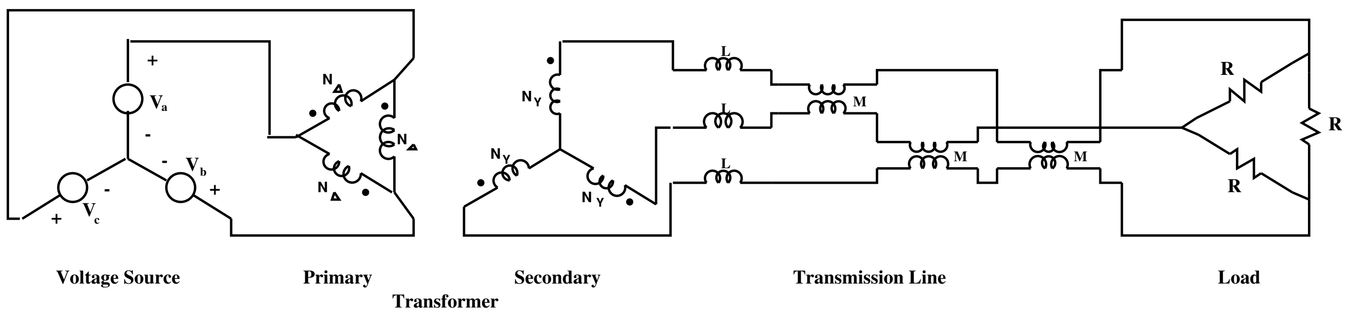

Figure 21: Example

complex amplitudes, it is possible to write the voltage drops for the three phases by:

$$\underline{V}_{a1} - \underline{V}_{a2} = j\omega L \underline{I}_a + j\omega M \left(\underline{I}_b + \underline{I}_c \right) \tag{43}$$
$$\underline{V}_{b1} - \underline{V}_{b2} = j\omega L \underline{I}_b + j\omega M \left(\underline{I}_a + \underline{I}_c \right) \tag{44}$$
$$\underline{V}_{c1} - \underline{V}_{c2} = j\omega L \underline{I}_c + j\omega M \left(\underline{I}_a + \underline{I}_b \right) \tag{45}$$

If the currents form a balanced set:

$$\underline{I}_a + \underline{I}_b + \underline{I}_c = 0 \tag{46}$$

Then the voltage drops are simply:

$$\begin{aligned} \underline{V}_{a1} - \underline{V}_{a2} &= j\omega \left(L - M \right) \underline{I}_a \\ \underline{V}_{b1} - \underline{V}_{b2} &= j\omega \left(L - M \right) \underline{I}_b \\ \underline{V}_{c1} - \underline{V}_{c2} &= j\omega \left(L - M \right) \underline{I}_c \end{aligned}$$

In this case, an apparent inductance, suitable for the balanced case, has been defined:

$$L_1 = L - M \tag{47}$$

which describes the behavior of one phase in terms of its own current. It is most important to note that this inductance is a valid description of the line only if (46) holds, which it does, of course, in the *balanced* case.

7.1 Example

To show how the analytical techniques which come from the network simplification resulting from single phase equivalents and wye-delta transformations, consider the following problem:

> A three-phase resistive load is connected to a balanced three-phase source through a transformer connected in *delta-wye* and a polyphase line, as shown in Figure 21. The problem is to calculate power dissipated in the load resistors. The three- phase voltage source has:

$$\begin{aligned} v_a &= Re \left[\sqrt{2} V_{RMS} e^{j\omega t} \right] \\ v_b &= Re \left[\sqrt{2} V_{RMS} e^{j\left(\omega t - \frac{2\pi}{3}\right)} \right] \\ v_c &= Re \left[\sqrt{2} V_{RMS} e^{j\left(\omega t + \frac{2\pi}{3}\right)} \right] \end{aligned}$$

This problem is worked by a succession of simple transformations. First, the *delta* connected resistive load is converted to its equivalent *wye* with $R_Y = \frac{R}{3}$.

Next, since the problem is balanced, the self- and mutual inductances of the line are directly equivalent to self inductances in each phase of $L_1 = L - M$.

Now, the transformer secondary is facing an impedance in each phase of:

$$\underline{Z}_{Ys} = j\omega L_1 + R_Y$$

The *delta-wye* transformer has a *voltage* ratio of:

$$\frac{v_p}{v_s} = \frac{N_\Delta}{\sqrt{3}N_Y}$$

so that, on the primary side of the transformer, the line and load impedance is:

$$\underline{Z}_p = j\omega L_{eq} + R_{eq}$$

where the equivalent elements are:

$$\begin{aligned} L_{eq} &= \frac{1}{3}\left(\frac{N_\Delta}{N_Y}\right)^2 (L - M) \\ R_{eq} &= \frac{1}{3}\left(\frac{N_\Delta}{N_Y}\right)^2 \frac{R}{3} \end{aligned}$$

Magnitude of current flowing in each phase of the source is:

$$|\underline{I}| = \frac{\sqrt{2}V_{RMS}}{\sqrt{(\omega L_{eq})^2 + R_{eq}^2}}$$

Dissipation in one phase is:

$$\begin{aligned} P_1 &= \frac{1}{2}|\underline{I}|^2 R_{eq} \\ &= \frac{V_{RMS}^2 R_{eq}}{(\omega L_{eq})^2 + R_{eq}^2} \end{aligned}$$

And, of course, total power dissipated is just three times the single phase dissipation.

8 Introduction To Per-Unit Systems

Strictly speaking, *per-unit* systems are nothing more than normalizations of voltage, current, impedance and power. These normalizations of system parameters because they provide simplifications in many network calculations. As we will discover, while certain *ordinary* parameters have very wide ranges of value, the equivalent *per-unit* parameters fall in a much narrower range. This helps in understanding how certain types of system behave.

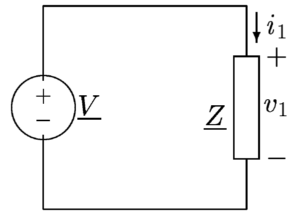

Figure 22: Example

8.1 Normalization Of Voltage And Current

The basis for the per-unit system of notation is the expression of voltage and current as fractions of *base* levels. Thus the first step in setting up a per-unit normalization is to pick *base* voltage and current.

Consider the simple situation shown in Figure 22. For this network, the complex amplitudes of voltage and current are:

$$\underline{V} = \underline{I}\underline{Z} \tag{48}$$

We start by defining two *base* quantities, V_B for voltage and I_B for current. In many cases, these will be chosen to be nominal or rated values. For generating plants, for example, it is common to use the rated voltage and rated current of the generator as *base* quantities. In other situations, such as system stability studies, it is common to use a standard, system wide base system.

The *per-unit* voltage and current are then simply:

$$\underline{v} = \frac{\underline{V}}{V_B} \tag{49}$$

$$\underline{i} = \frac{\underline{I}}{I_B} \tag{50}$$

Applying (49) and (50) to (48), we find:

$$\underline{v} = \underline{i}\underline{z} \tag{51}$$

where the *per-unit* impedance is:

$$\underline{z} = \underline{Z}\frac{I_B}{V_B} \tag{52}$$

This leads to a definition for a *base impedance* for the system:

$$Z_B = \frac{V_B}{I_B} \tag{53}$$

Of course there is also a *base power*, which for a single phase system is:

$$P_B = V_B I_B \tag{54}$$

as long as V_B and I_B are expressed in RMS. It is interesting to note that, as long as normalization is carried out in a consistent way, there is no ambiguity in per-unit notation. That is, *peak* quantities normalized to *peak* base quantities will be the same, in per-unit, as RMS quantities normalized to RMS bases. This advantage is even more striking in polyphase systems, as we are about to see.

8.2 Three Phase Systems

When describing polyphase systems, we have the choice of using either line-line or line-neutral voltage and line current or current in delta equivalent loads. In order to keep straight analysis in *ordinary* variable, it is necessary to carry along information about which of these quantities is being used. There is no such problem with *per-unit* notation.

We may use as base quantities either line to neutral voltage V_{Bl-g} or line to line voltage V_{Bl-l}. Taking the base *current* to be line current I_{Bl}, we may express base *power* as:

$$P_B = 3V_{Bl-g}I_{Bl} \tag{55}$$

Because line-line voltage is, under normal operation, $\sqrt{3}$ times line-neutral voltage, an equivalent statement is:

$$P_B = \sqrt{3}V_{Bl-l}I_{Bl} \tag{56}$$

If base *impedance* is expressed by line-neutral voltage and line current (This is the common convention, but is not required),

$$Z_B = \frac{V_{Bl-g}}{I_{Bl}} \tag{57}$$

Then, base impedance is, written in terms of base power:

$$Z_B = \frac{P_B}{3I_B^2} = 3\frac{V_{Bl-g}^2}{P_B} = \frac{V_{Bl-l}^2}{P_B} \tag{58}$$

Note that a single per-unit voltage applied equally well to line-line, line-neutral, peak and RMS quantities. For a given situation, each of these quantities will have a different *ordinary* value, but there is only one *per-unit* value.

8.3 Networks With Transformers

One of the most important advantages of the use of per-unit systems arises in the analysis of networks with transformers. Properly applied, a per-unit normalization will cause nearly all ideal transformers to dissapear from the per-unit network, thus greatly simplifying analysis.

To show how this comes about, consider the ideal transformer as shown in Figure 23. The

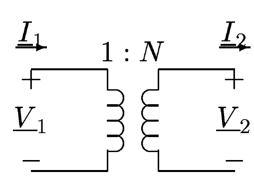

Figure 23: Ideal Transformer With Voltage And Current Conventions Noted

ideal transformer imposes the constraints that:

$$\begin{aligned} \underline{V}_2 &= N\underline{V}_1 \\ \underline{I}_2 &= \frac{1}{N}\underline{I}_1 \end{aligned}$$

Normalized to base quantities on the two sides of the transformer, the per-unit voltage and current are:

$$\begin{aligned} \underline{v}_1 &= \frac{\underline{V}_1}{V_{B1}} \\ \underline{i}_1 &= \frac{\underline{I}_1}{I_{B1}} \\ \underline{v}_2 &= \frac{\underline{V}_2}{V_{B2}} \\ \underline{i}_2 &= \frac{\underline{I}_2}{I_{B2}} \end{aligned}$$

Now: note that if the *base* quantities are related to each other as if *they* had been processed by the transformer:

$$V_{B2} = NV_{B1} \tag{59}$$

$$I_{B2} = \frac{I_{B1}}{N} \tag{60}$$

then $\underline{v}_1 = \underline{v}_2$ and $\underline{i}_1 = \underline{i}_2$, as if the ideal transformer were not there (that is, consisted of an ideal wire).

Expressions (59) and (60) reflect a general rule in setting up per-unit normalizations for systems with transformers. Each segment of the system should have the same base *power*. Base *voltages* transform according to transformer *voltage* ratios. For three-phase systems, of course, the *voltage* ratios may differ from the physical turns ratios by a factor of $\sqrt{3}$ if *delta-wye* or *wye-delta* connections are used. It is, however, the *voltage* ratio that must be used in setting base voltages.

8.4 Transforming From One Base To Another

Very often data such as transformer leakage inductance is given in per-unit terms, on some base (perhaps the units rating), while in order to do a system study it is necessary to express the same data in per-unit in some other base (perhaps a unified system base). It is always possible to do this by the two step process of converting the per-unit data to its *ordinary* form, then re-normalizing it in the new base. However, it is easier to just convert it to the new base in the following way.

Note that impedance in Ohms (*ordinary* units) is given by:

$$\underline{Z} = \underline{z}_1 Z_{B1} = \underline{z}_2 Z_{B2} \tag{61}$$

Here, of course, $\underline{z}_1$ and $\underline{z}_2$ are the same *per-unit* impedance expressed in different *bases*. This could be written as:

$$\underline{z}_1 \frac{V_{B1}^2}{P_{B1}} = \underline{z}_2 \frac{V_{B2}^2}{P_{B2}} \tag{62}$$

This yields a convenient rule for converting from one base system to another:

$$\underline{z}_1 = \frac{P_{B1}}{P_{B2}} \left(\frac{V_{B2}}{V_{B1}} \right)^2 \underline{z}_2 \tag{63}$$

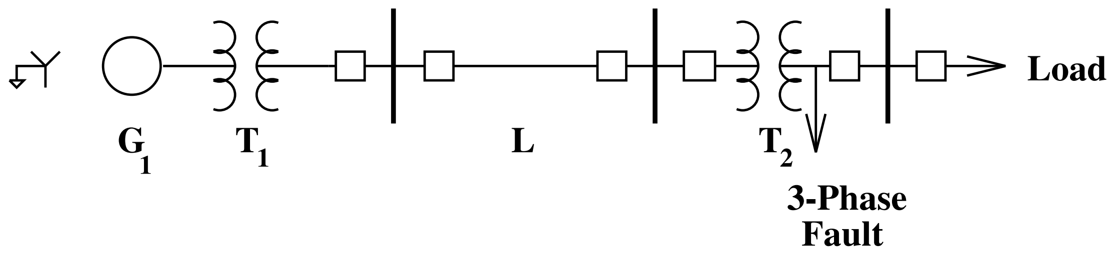

Figure 24: One-Line Diagram Of Faulted System

8.5 Example: Fault Study

To illustrate some of the concepts with which we have been dealing, we will do a short circuit analysis of a simple power system. This system is illustrated, in one-line diagram form, in Figure 24.

A one-line diagram is a way of conveying a lot of information about a power system without becoming cluttered with repetitive pieces of data. Drawing all three phases of a system would involve quite a lot of repetition that is not needed for most studies. Further, the three phases *can* be re-constructed from the one-line diagram if necessary. It is usual to use special symbols for different components of the network. For our network, we have the following pieces of data:

Symbol	Component	Base P (MVA)	Base V (kV)	Impedance (per-unit)
G_1	Generator	200	13.8	$j.18$
T_1	Transformer	200	13.8/138	$j.12$
L_1	Trans. Line	100	138	$.02 + j.05$
T_2	Transformer	50	138/34.5	$j.08$

A three-phase fault is assumed to occur on the 34.5 kV side of the transformer T_2. This is a symmetrical situation, so that only one phase must be represented. The per-unit impedance diagram is shown in Figure 25. It is necessary to proceed now to determine the value of the components in this circuit.

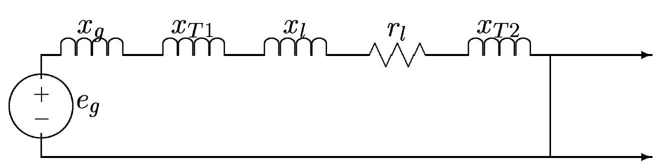

Figure 25: Impedance Diagram For Fault Example

First, it is necessary to establish a uniform base an per-unit value for each of the system components. Somewhat arbitrarily, we choose as the base segment the transmission line. Thus all of the parameters must be put into a *base power* of 100 MVA and voltage bases of 138 kV on the line, 13.8 kV at the generator, and 34.5 kV at the fault. Using (62):

$$\begin{aligned} x_g &= \tfrac{100}{200} \times .18 &= .09\text{per-unit} \\ x_{T1} &= \tfrac{100}{200} \times .12 &= .06\text{per-unit} \end{aligned}$$

$$
\begin{aligned}
x_{T2} = & \tfrac{100}{50} \times .08 & = .16 \text{per-unit} \\
r_l = & & = .02 \text{per-unit} \\
x_l = & & = .05 \text{per-unit}
\end{aligned}
$$

Total impedance is:

$$
\begin{aligned}
\underline{z} &= j\left(x_g + x_{T1} + x_l + x_{T2}\right) + r_l \\
&= j.36 + .02 \text{per-unit} \\
|\underline{z}| &= .361 \text{per-unit}
\end{aligned}
$$

Now, if e_g is equal to one per-unit (generator internal voltage equal to base voltage), then the per-unit *current* is:

$$
|\underline{i}| = \frac{1}{.361} = .277 \text{per-unit}
$$

This may be translated back into ordinary units by getting base current levels. These are:

- On the base at the generator:

$$
I_B = \frac{100\text{MVA}}{\sqrt{3} \times 13.8\text{kV}} = 4.18\text{kA}
$$

- On the line base:

$$
I_B = \frac{100\text{MVA}}{\sqrt{3} \times 138\text{kV}} = 418\text{A}
$$

- On the base at the fault:

$$
I_B = \frac{100\text{MVA}}{\sqrt{3} \times 34.5\text{kV}} = 1.67\text{kA}
$$

Then the actual fault currents are:

- At the generator $|\underline{I}_f| = 11,595\text{A}$
- On the transmission line $|\underline{I}_f| = 1159\text{A}$
- At the fault $|\underline{I}_f| = 4633\text{A}$