





Class Notes for  
Modern Optics Project Laboratory - 6.161  
Optical Signals, Devices and Systems - 6.637  
Basic Properties of Electromagnetic Waves

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September 11, 2007



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# Chapter 1

## BASIC PROPERTIES OF ELECTROMAGNETIC WAVES

### 1.1 INTRODUCTION

The optical region of the electromagnetic spectrum is loosely defined as the region spanned by the ultraviolet, visible and the infrared. The complete electromagnetic spectrum is shown below in Fig. 1.1. Regarding the units in the Table,  $1 \mu\text{m} = 10^{-6} \text{ m}$ ;  $1 \text{ nm} = 10^{-9} \text{ m}$ ;  $1 \text{ A} = 10^{-10} \text{ m}$ ;  $1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$ ;  $h = 6.625 \times 10^{-34} \text{ J.s}$ .

	wave region				← optical region →			ray region	
	wireless	radio broadcast	short radio	micro wave	infrared	visible	ultra violet	x-ray	$\gamma$ -ray
$\lambda$	1000m	100m	30cm	300 $\mu\text{m}$	700nm	400nm	50 $\text{\AA}$	0.07 $\text{\AA}$	
$\nu$	$3 \times 10^5$	$3 \times 10^6$	$10^9$	$10^{12}$	$4.3 \times 10^{14}$	$7.5 \times 10^{14}$	$6 \times 10^{16}$	$4.3 \times 10^{19}$ (Hz)	
E	$1.2 \times 10^{-9}$	$1.2 \times 10^{-8}$	$4 \times 10^{-6}$	0.004	1.7	2.3	248	$1.8 \times 10^5$ (eV)	

Figure 1.1: The electromagnetic spectrum

Electromagnetic waves are coupled oscillatory electric and magnetic fields ( $\mathbf{E}$  and  $\mathbf{H}$  respectively) that propagate in media as an energy flow through space and time. The spatial and temporal relationships between the electric and magnetic field components, oscillating at frequency  $\omega$ , are governed by Maxwell's equations, which in MKS units are as follows:

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t) \quad (1.1)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D}(\mathbf{r}, t) \quad (1.2)$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho \quad (1.3)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \quad (1.4)$$

where  $\mathbf{E}$  and  $\mathbf{H}$  are the electric and magnetic field strengths in Volts/m and Amperes/m respectively,  $\mathbf{D}$  is the electric displacement in Coulombs/m<sup>2</sup>,  $\mathbf{B}$  is the magnetic flux density in Weber/m<sup>2</sup> or Tesla,  $\mathbf{J}$  is the conduction and convection current density in Amperes/m<sup>2</sup>,  $\rho$  is the free charge density in Coulombs/m<sup>3</sup>,  $\mathbf{r}$  is the spatial coordinate and  $t$  is time. Here, boldface characters are used to indicate that the quantities are vectors.

In a medium (material) characterized by permittivity  $\epsilon$ , permeability  $\mu$ , and conductivity  $\sigma$ , each of which are frequency dependent, the following constitutive relations hold:

$$\mathbf{D} = \epsilon \mathbf{E} \quad (1.5)$$

$$\mathbf{B} = \mu \mathbf{H}$$

$$\mathbf{J} = \sigma \mathbf{E} \quad (\text{Ohm's law}) \quad (1.6)$$

The units of conductivity,  $\sigma$ , are (ohm.m)<sup>-1</sup> or Siemens. An electric field acting on a material will polarize that material. Some materials are anisotropic. That is, the induced polarization, for the same applied electric field, will be different along different directions within the material. In such a case, the permittivity,  $\epsilon$ , will have to be represented by a tensor. The relationship between  $\mathbf{D}$ ,  $\mathbf{E}$  and the polarization  $\mathbf{P}$  (dipole moment per unit volume) of the material is

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (1.7)$$

where  $\epsilon_0$  is the permittivity of free space. Thus we see that

$$\epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (1.8)$$

and that the relative permittivity (dielectric constant),  $\epsilon/\epsilon_0$ , of the material, which is frequency dependent, is given by

$$\epsilon(\omega)/\epsilon_0 = 1 + \mathbf{P}/\epsilon_0 \mathbf{E} \quad (1.9)$$

For small values of the electric field (small compared with inter-atomic forces), the electric polarization  $\mathbf{P}$  is linear in  $\mathbf{E}$  and can be written as

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} \quad (1.10)$$

where  $\chi$  is called the electric susceptibility. Hence,

$$\epsilon(\omega)/\epsilon_0 = 1 + \chi(\omega) \quad (1.11)$$

In an anisotropic material, the induced polarization for the same  $\mathbf{E}$  field applied along different directions of the material will be different in magnitude and direction, and so, in general,  $\mathbf{D}$  differs from  $\mathbf{E}$  in such materials. In general  $\epsilon$  and  $\chi$  are both tensors.

Similarly, for the magnetic field, both the magnetic field  $\mathbf{H}$  and the magnetization  $\mathbf{M}$  of the material contribute to the magnetic flux density  $\mathbf{B}$ , but the equation is generally written slightly differently from its electric counterpart, as

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) \quad (1.12)$$

where  $\mu_0$  is the permeability of free space. We note that  $\epsilon_0 = 8.85 \times 10^{-12}$  F/m, and  $\mu_0 = 4\pi \times 10^{-7}$  H/m.

By vector algebraic manipulation of Eqns. 1.1-1.6, we can derive the following wave equations which describe the propagation characteristics of electromagnetic waves in a medium characterized by  $\epsilon$ ,  $\mu$ ,  $\sigma$  and  $\rho$ :

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} - \nabla \left( \frac{\rho}{\epsilon} \right) = 0 \quad (1.13)$$

and

$$\nabla^2 \mathbf{H} - \epsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} - \sigma\mu \frac{\partial \mathbf{H}}{\partial t} = 0 \quad (1.14)$$

Note that there is no magnetic equivalent of free electric charge, and hence no fourth term in the  $\mathbf{H}$  wave equation. In non-conductors, which are characterized by  $\sigma = 0$ , if  $\rho = 0$  within the material, the wave equation simplifies to

$$\nabla^2 \mathbf{E}(\mathbf{r}, t) - \epsilon\mu \frac{\partial^2 \mathbf{E}(\mathbf{r}, t)}{\partial t^2} = 0 \quad (1.15)$$

where the wave velocity  $v$  is given by  $v = (\epsilon\mu)^{-1/2}$ . The index of refraction  $n$  of the medium is defined by the ratio of the speed of light in vacuum to the speed of light in the medium. That is,

$$n = \frac{c}{v} = \frac{(\epsilon\mu)^{1/2}}{(\epsilon_0\mu_0)^{1/2}} = (K_e K_m)^{1/2} \quad (1.16)$$

where  $K_e = \epsilon/\epsilon_0$  is the relative permittivity (or the dielectric constant), and  $K_m = \mu/\mu_0$  is the relative permeability. In charge-neutral insulators ( $\mu \simeq \mu_0$ ,  $K_m \simeq 1$ ), the refractive index  $n$  is simply the square root of the dielectric constant at the frequency of the wave.<sup>1</sup> That is,  $n = (K_e)^{1/2}$ . For example, for glass,  $n \simeq 1.5$  and for water,  $n \simeq 1.3$ . In general,  $n = n(\omega)$ , where  $\omega$  is the angular frequency of the wave. This frequency-dependent property of the refractive index is called dispersion.

The general solution to the wave equation can be written in complex amplitude (phasor) notation as

$$\mathbf{E}(\mathbf{r}, t) = \Re \left\{ \underline{\mathbf{E}}(\mathbf{r}, t) e^{-j\omega t} \right\} \quad (1.17)$$

Here the underbar on the boldface field quantity signifies that it is a complex function. In particular, for monochromatic light, there can be no temporal modulation on  $\underline{\mathbf{E}}(\mathbf{r}, t)$  (e.g., amplitude modulation) since this would broaden the spectrum of  $\underline{\mathbf{E}}(\mathbf{r}, t)$  in accordance with the uncertainty principle ( $\Delta\omega \cdot \Delta t \sim 2\pi$ ). Thus, for a monochromatic plane wave, the phasor  $\underline{\mathbf{E}}(\mathbf{r}, t)$  must be time-independent. For Eqn. 1.15, we can postulate a monochromatic *plane wave* solution where the phasor  $\underline{\mathbf{E}}(\mathbf{r}, t)$  is of the form

$$\underline{\mathbf{E}}(\mathbf{r}, t) = \underline{\mathbf{E}}(\mathbf{r}) = \hat{e} \underline{E}_0 e^{j\mathbf{k} \cdot \mathbf{r}} = \hat{e} (E_0 e^{j\phi}) e^{j\mathbf{k} \cdot \mathbf{r}} \quad (1.18)$$

---

<sup>1</sup>  $\epsilon/\epsilon_0 \approx n^2$  is true only at optical frequencies. At low frequencies (e.g., at DC),  $\epsilon/\epsilon_0$  has contributions from the lattice and is generally larger than at optical frequencies.

where  $\mathbf{k}$  is the *propagation vector*,  $\phi$  is a reference phase, and  $\hat{e}$  is a unit vector that captures the polarization properties of the field. The unit vector  $\hat{e}$  is called the *polarization vector*. By convention the polarization of the light is defined as the direction of oscillation of the electric field of the wave; and as you can see, it is the vectorial property of the wave.

Thus, the full monochromatic plane wave solution for a wave traveling in a medium characterized by  $\epsilon$  and  $\mu$  is

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= \Re\{ \hat{e} E_0 e^{j(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)} \} \\ &= \hat{e} E_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi).\end{aligned}\tag{1.19}$$

It is straightforward to show that in order for the above postulated plane monochromatic wave solution to satisfy the source-free wave equation for non-conducting media,  $k$ , the magnitude of the propagation vector, (also called the wave number) must be

$$k = \omega(\mu\epsilon)^{1/2} = \omega/v = 2\pi c/\lambda v = 2\pi n/\lambda\tag{1.20}$$

where  $\lambda$  is the wavelength of the light such that the velocity  $v$  of the wave is  $v = c/n = \lambda\nu$ , and  $\nu = \omega/2\pi$  is the frequency of the wave.

A similar set of equations hold for the  $\mathbf{H}$  field. In practice, it is sufficient to work with only one of the field quantities  $\mathbf{E}$  or  $\mathbf{H}$ , since for plane waves the two are related by the expression

$$\mathbf{H}(\mathbf{r}, t) = (1/\eta)\hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)\tag{1.21}$$

where  $\eta = (\mu/\epsilon)^{1/2}$  is the impedance of the medium, and  $\hat{\mathbf{k}}$  is the unit  $\mathbf{k}$  vector. In free space,  $\eta = \eta_0 = 377$  ohms. The above equation is a manifestation of the fact that the electromagnetic wave is a transverse wave in the sense that  $\mathbf{E}$  and  $\mathbf{H}$  are perpendicular to  $\mathbf{k}$ . Fig. 1.2 illustrates the oscillatory form of  $\mathbf{E}$  and  $\mathbf{H}$  in space at  $t = 0$  in accordance with Eqns. 1.19 and 1.21.

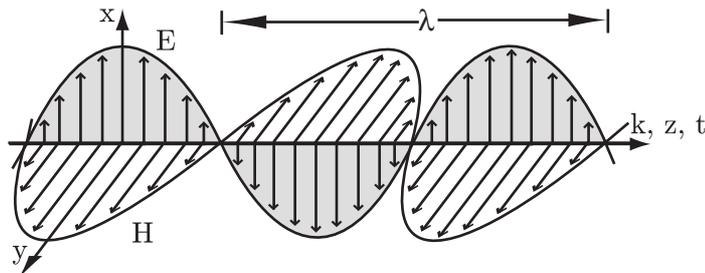


Figure 1.2: Illustrating the relative oscillation of the  $\mathbf{E}$  and  $\mathbf{H}$  fields in space, and time, and showing that  $\mathbf{E}$ ,  $\mathbf{H}$  and  $\mathbf{k}$  are mutually orthogonal.

Equation. 1.19 represents a plane wave because the wavefronts (surfaces of constant phase defined by  $(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi = \text{constant})$  are planar. Assuming the reference phase  $\phi$  is a constant, we can see from Fig. 1.3(a), that at some fixed time  $t_1$ , the set of points for which  $\mathbf{k}\cdot\mathbf{r} - \omega t_1$  is constant (equal to  $(2\pi/\lambda)OA$  in this case) lie on a plane. Similarly, to see how a given wavefront moves in time, we can let time increase and follow how  $\mathbf{r}$  must change so that

$\mathbf{k} \cdot \mathbf{r} - \omega t = \text{constant}$ . We see that  $\mathbf{r}$  must also increase, as illustrated in Fig. 1.3(a) for times  $t_1$ ,  $t_2$  and  $t_3$ .

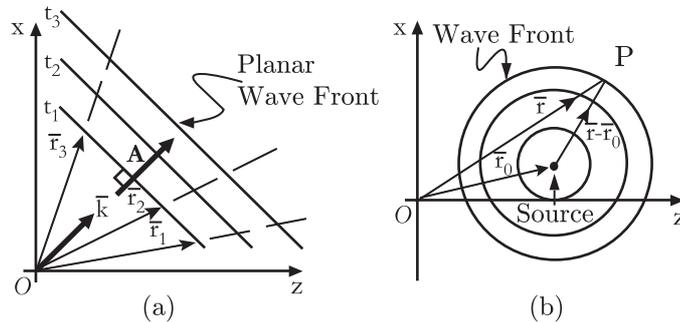


Figure 1.3: Co-ordinate system illustrating the propagation of (a) a plane wave, (b) a spherical wave

In addition to the plane monochromatic wave solution given above, there is also a spherical wave solution. That is, a wave whose wavefronts are parts of spheres. This solution can be written again as Eqn 1.17, but this time with

$$\underline{\mathbf{E}}(\mathbf{r}) = \frac{\underline{A}e^{\pm jk|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} \quad (1.22)$$

where  $\underline{A}$  is a complex constant given by  $\underline{A} = Ae^{j\phi}$ ,  $k^2 = \omega^2\epsilon\mu$  as before,  $\mathbf{r}_0$  is the location of the source, and  $\mathbf{r}$  is the vector to the observation point,  $P$  as defined in Fig. 1.3(b) for an outward propagating beam. The plus sign in the exponent corresponds to a wave diverging from the point  $\mathbf{r} = \mathbf{r}_0$  and the minus sign to a wave converging onto the point  $\mathbf{r} = \mathbf{r}_0$ . This can be seen by noting that each surface of constant phase (spherical wavefront in this case) must satisfy an equation of motion of the form

$$\pm k|\mathbf{r}-\mathbf{r}_0| - \omega t = K \quad (1.23)$$

where  $K$  is a constant. Thus if the plus sign is chosen, as  $t$  increases  $|r|$  must increase (diverging wave).

## 1.2 ENERGY FLOW

The instantaneous stored energy density,  $u$ , of the electromagnetic wave is made up equally of electric and magnetic components. Thus, in general, we have

$$u = u_E + u_H = (1/2)[\epsilon E^2 + \mu H^2] = \epsilon E^2 = \mu H^2 \left[ \frac{\text{Joules}}{m^3} \right] \quad (1.24)$$

by virtue of Eqn.1.21. For isotropic media, the energy flow is in the direction of the propagation vector, and the magnitude  $S$  of the power flow per unit crosssectional area of the beam is

$$S = uc \left[ \frac{\text{Watts}}{m^2} \right] \quad (1.25)$$

The corresponding vector quantity  $\mathbf{S}(t)$  (also called the Poynting vector) for a plane monochromatic wave is given by

$$\begin{aligned} \mathbf{S}(t) &= \mathbf{E}(t) \times \mathbf{H}(t) = \hat{k} |E(t)|^2 / \eta \\ &= \hat{k} (E_0^2 / \eta) \cos^2(\mathbf{k} \cdot \mathbf{r} - \omega t) \end{aligned} \quad (1.26)$$

The intensity  $I$  of the wave is defined by the time average of the Poynting vector. That is,

$$I(\mathbf{r}) = \langle \mathbf{S}(\mathbf{r}, t) \rangle = \langle \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \rangle = \frac{1}{\eta} \langle |\mathbf{E}(\mathbf{r}, t)|^2 \rangle = E_0^2 / 2\eta \quad (1.27)$$

where the angle brackets denote time averaging. Note that in free space the intensity of a plane wave can be rewritten as

$$I = \langle \mathbf{S}(t) \rangle = c\epsilon_0 E_0^2 / 2 \quad (1.28)$$

It can also be shown that the intensity  $I(\mathbf{r}) = \frac{1}{2} \Re\{\mathbf{S}\}$  where  $\mathbf{S}$  is related to the phasors  $\underline{\mathbf{E}}$  and  $\underline{\mathbf{H}}$  by  $\mathbf{S} = \underline{\mathbf{E}} \times \underline{\mathbf{H}}^*$

## 1.3 POLARIZATION STATES OF THE ELECTROMAGNETIC FIELD

As you might expect, a plane electromagnetic wave with a general state of polarization can be written as an admixture of two orthogonal linearly polarized plane waves (with phasors  $\underline{\mathbf{E}}_1(\mathbf{r}) = \hat{e}_1 E_1 e^{j\phi_1}$  and  $\underline{\mathbf{E}}_2(\mathbf{r}) = \hat{e}_2 E_2 e^{j\phi_2}$ , as

$$\mathbf{E}(\mathbf{r}, t) = \Re \left\{ \left[ \hat{e}_1 E_1 e^{j\phi_1} + \hat{e}_2 E_2 e^{j\phi_2} \right] e^{j(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right\} \quad (1.29)$$

where  $\hat{e}_1$  and  $\hat{e}_2$  are orthogonal polarization vectors. There are two special cases of light polarization: (1) linear polarized light, and (2) circular polarized light. Elliptical polarization is the most general case.

### 1.3.1 Linear Polarization

If the electric field vector always oscillates in the same direction, the wave is said to be linearly polarized. For example, a plane wave traveling in the  $z$ -direction and polarized along the  $x$ -direction may be written out explicitly as

$$\mathbf{E}(z, t) = \Re \left\{ \hat{x} E_0 e^{j(kz - \omega t + \phi)} \right\} \quad (1.30)$$

since  $\underline{E}_0$  is of the form  $\underline{E}_0 = E_0 e^{j\phi}$ . Thus a linear polarized wave is the special case of Eqn. 1.29 that occurs when  $\underline{\mathbf{E}}_1$  and  $\underline{\mathbf{E}}_2$  have the same phase ( $\phi_1 = \phi_2 = \phi$ ).

Linear polarized light is produced by passing light through a linear polarizer. A typical linear polarizer consists of a material that absorbs the component of light which is polarized along one of the principal axes of the material, and passes the orthogonal polarization without absorption. An example is an aligned array of needle-shaped molecules that absorb light, when the electric field vector of the light is oriented, say, along the long molecular axis and passes light when the electric field vector is perpendicular to the long molecular axis (e.g., iodine molecules which are sometimes used in polarizers for visible light). In the infrared, the same phenomenon is achieved with a fine wire grating. These so-called wire-grid polarizers absorb light for which the electric field is oriented along the axis of the wires. This is because this component of the electric field drives the electrons in the wire to produce oscillating currents, but since the wire has finite resistance, joule heating leads to absorption (as opposed to re-radiation) of this component of the electric field.

### 1.3.2 Circular Polarization

In the special case when  $\underline{\mathbf{E}}_1$  and  $\underline{\mathbf{E}}_2$  in Eqn. 1.29 have the same magnitude, but differ in phase by  $90^\circ$ , the wave is circularly polarized, and the tip of the electric field vector traces out a circle in space for every wavelength advance of the beam.

Based on the foregoing, we can describe the behavior of the electric field of a circularly polarized wave propagating in the  $+z$  direction by

$$\mathbf{E}(z, t) = \Re \left\{ E_0 (\hat{x} \pm j\hat{y}) e^{j(kz - \omega t)} \right\} \quad (1.31)$$

since  $j = e^{j\frac{\pi}{2}}$ . Thus for the plus sign, we have the decomposition:

$$E_x(z, t) = E_0 \cos(kz - \omega t) \quad (1.32)$$

and

$$E_y(z, t) = -E_0 \sin(kz - \omega t) \quad (1.33)$$

Thus the total electric field amplitude  $E$  is

$$E = (E_x^2 + E_y^2)^{1/2} = E_0 \quad (1.34)$$

We also see that at a fixed point in space, the field components are such that the total electric vector is constant in magnitude and its tip sweeps out a circle at frequency  $\omega$ . In particular, for the plus sign [i.e.,  $(\hat{e}_1 + j\hat{e}_2)$  in Eqn. 1.31] the rotation is counter-clockwise when the observer is facing the oncoming wave, and the wave is right-hand circularly polarized. This can be seen from Eqn. 1.31 by drawing the components  $E_x$  and  $E_y$  in the  $z = 0$  plane for increasing values of  $t$  (e.g.,  $\omega t = 0, \pi/2, \pi, \dots$ ) as shown in Fig. 1.4.

With the thumb pointing in the direction of propagation, as indicated in Fig. 1.4, the right hand is needed to have the other fingers point in the direction of rotation of the electric field. Thus, this corresponds to right-handed, circularly-polarized (RHCP) light.

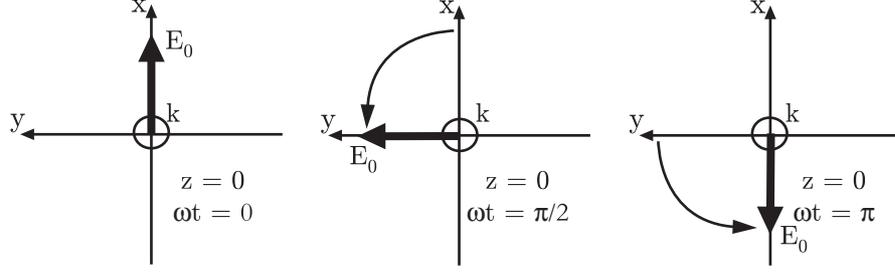


Figure 1.4: Sketch of the polarization vector in the  $z = 0$  plane (based on Eqns. 1.32 and 1.33) for  $\omega t = 0, \pi/2, \pi$  for the plus sign in Eqn. 1.31. Note that light is propagating out of the page toward the reader. This corresponds to right-handed circularly polarized light.

### 1.3.3 Elliptical Polarization

Elliptical polarization occurs when  $\underline{E}_1$  and  $\underline{E}_2$  in Eqn. 1.29 have different phases. In this case the tip of the resultant electric field vector traces out an ellipse in space each time the wave propagates one wavelength.

The conditions leading to the various states of polarization are summarized in Table I. Here we rewrite the general state of polarization (admixture of an  $x$ -polarized and a  $y$ -polarized wave) for propagation along the  $z$  axis as

$$\underline{E}(z, t) = \hat{x}E_x \cos(kz - \omega t) + \hat{y}E_y \cos(kz - \omega t + \Delta\phi) \quad (1.35)$$

where  $\Delta\phi = \phi_y - \phi_x$ .

Case	Condition	Polarization	Amplitude
1	$E_y = 0$	Linear along $x$	$E_x$
2	$E_x = 0$	Linear along $y$	$E_y$
3	$E_x = E_y = E_0; \Delta\phi = 0$	Linear at $45^\circ$ to axis	$\sqrt{2}E_0$
4	$E_x \neq E_y; \Delta\phi = 0$	Linear at $\tan^{-1}(E_y/E_x)$ to $x$ axis	$\sqrt{(E_x^2 + E_y^2)}$
5	$E_x = E_y = E_0; \Delta\phi = \pi/2$	Right hand circular	$E_0$
6	$E_x = E_y = E_0; \Delta\phi = -\pi/2$	Left hand circular	$E_0$
7	Everything Else	Elliptical	time varying

**Example 1:** What is the polarization state corresponding to  $E_x \neq E_y; \Delta\phi \neq 0$ ?

*Solution:* (a) for  $-\pi < \Delta\phi < 0$ , left hand elliptical; (b) for  $0 < \Delta\phi < \pi$ , right hand elliptical.

**Example 2:** Find the direction of propagation and describe the polarization for light defined by the phasor

$$\underline{E}(\vec{r}) = (\hat{x} + \sqrt{3}\hat{y})e^{-jkz}$$

*Solution:* Propagation is in the  $-z$  direction;  $E_x = 1; E_y = \sqrt{3}; \Delta\phi = 0$ , hence this is a linearly polarized wave,  $\tan\psi = \sqrt{3}$ ; that is  $\psi = 60^\circ$  where  $\psi$  is the angle between the direction of oscillation of the  $E$  field and the  $x$  axis.

### 1.3.4 Producing circular polarized light

A quarter-wave plate is an optical element that can be used to produce circular polarized light. It is made of optically anisotropic material that has two principal axes of propagation: a fast axis with index of refraction  $n_f$  and a slow axis with index of refraction  $n_s$ . Light polarized along the fast axis travels at a speed given by  $c/n_f$ , and similarly light polarized along the slow axis travels at a speed of  $c/n_s$ . The thickness,  $d$ , of the plate is such that for in-phase fast and slow waves entering the plate, their differential phase shift,  $\phi_s - \phi_f$  after propagating through the plate is an odd multiple of  $\pi/2$  radians (quarter wave). That is,

$$\phi_s - \phi_f = 2\pi(n_s - n_f)d/\lambda = (2m + 1)\pi/2 \quad (1.36)$$

or

$$d = \frac{(2m + 1)\lambda}{4(n_s - n_f)} \quad (1.37)$$

where  $m$  is an integer.

Note that for a given material with fixed  $n_s$  and  $n_f$ , a  $\lambda/4$  plate is wavelength specific. The optical system shown in Fig. 1.5 produces circularly polarized light from randomly polarized light. The axis of the linear polarizer is oriented at  $45^\circ$  to the fast and slow axes of the  $\lambda/4$  plate. This guarantees that the amplitudes of  $\mathbf{E}_s$  and  $\mathbf{E}_f$  are equal at the input surface to the  $\lambda/4$  plate. The phase difference  $\phi_s - \phi_f = +\pi/2, +3\pi/2, \dots$  is adjusted by the thickness of the  $\lambda/4$  plate in accordance with Eqn. 1.37.

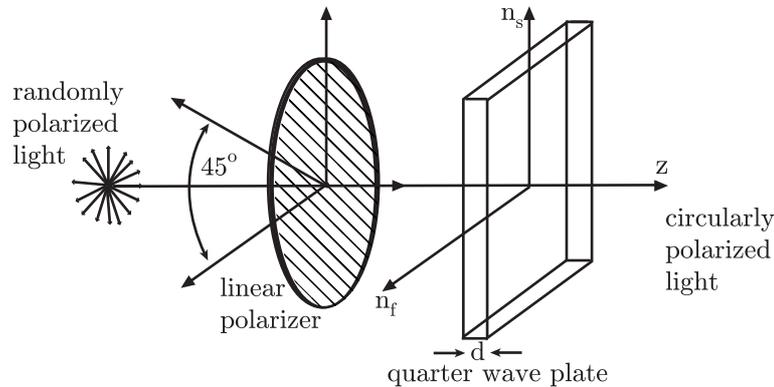


Figure 1.5: Setup for producing circularly polarized light

## 1.4 JONES VECTOR REPRESENTATION OF POLARIZATION

The Jones vector is a two-component vector representation of the complex electric field (phasor) of the light. Thus, for a monochromatic plane wave propagating in the  $z$  direction with the most general form of polarization

$$\mathbf{E}(z, t) = \Re \left\{ \left( \hat{x} E_x e^{j\phi_x} + \hat{y} E_y e^{j\phi_y} \right) e^{j(kz - \omega t)} \right\} \quad (1.38)$$

the unnormalized Jones vector representation for the field phasor would be

$$\underline{\mathbf{E}} = \begin{bmatrix} E_x e^{j\phi_x} \\ E_y e^{j\phi_y} \end{bmatrix} \quad (1.39)$$

So for linear polarization along the  $x$  axis, the Jones vector would be of the form

$$\underline{\mathbf{E}} = \begin{bmatrix} E_x e^{j\phi_x} \\ 0 \end{bmatrix} = E_0 e^{j\phi_0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (1.40)$$

with  $E_x = E_0$  and  $\phi_x = \phi_0$ . The Jones vector representation then simplifies to

$$\underline{\mathbf{E}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (1.41)$$

Similarly, for light linearly polarized at  $+45^\circ$  to the  $x$  axis for which  $E_x = E_y = E_0$  and  $\phi_x = \phi_y = \phi_0$ , the Jones vector would be

$$\underline{\mathbf{E}}_{45} = E_0 e^{j\phi_0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ or simply } \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (1.42)$$

In many applications, knowledge of the magnitudes and absolute phases of the components is unnecessary. In these cases, the convention is to work with the normalized form of the Jones vector. The normalization is in intensity, so the normalized Jones vector becomes

$$\underline{\mathbf{E}}_{45} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (1.43)$$

For right-hand circular polarized light,  $E_x = E_y = E_0$ , and the  $y$  component lags the  $x$  component by  $90^\circ$ . That is  $\phi_y = \phi_x + 90^\circ$ . Thus the Jones vector is

$$\underline{\mathbf{E}}_{RC} = E_0 e^{j\phi_x} \begin{bmatrix} 1 \\ e^{j\pi/2} \end{bmatrix} \quad (1.44)$$

and from Eqns. 1.31 and 1.34 we know that the normalized version of right-hand circular polarized light is

$$\underline{\mathbf{E}}_{RC} = \begin{bmatrix} 1 \\ j \end{bmatrix} \quad (1.45)$$

For elliptical polarized light,  $\phi_x$  and  $\phi_y$  have different phases, and  $E_x$  and  $E_y$  may have different magnitudes, but the same approach is used. For example, the left-handed elliptically polarized wave given by

$$\mathbf{E}(z, t) = \Re \left\{ \left( \hat{x} 2e^{j\pi} + \hat{y} 3e^{j\pi/4} \right) e^{j(kz - \omega t)} \right\} \quad (1.46)$$

would, factoring out the common phase term, be described by the normalized Jones vector

$$\underline{\mathbf{E}}_{LE} = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3e^{-j3\pi/4} \end{bmatrix} \quad (1.47)$$

Two complex vectors (phasors)  $\underline{E}_1$  and  $\underline{E}_2$  are orthogonal if  $\underline{E}_1 \cdot \underline{E}_2^* = 0$  where the  $*$  denotes conjugation. Thus we see that left and right-hand circular polarizations are orthogonal since

$$\underline{\mathbf{E}}_{LC} \cdot \underline{\mathbf{E}}_{RC}^* = [1 \quad -j] \begin{bmatrix} 1 \\ -j \end{bmatrix} = 0 \quad (1.48)$$

Some optical elements can convert one form of polarization  $\underline{\mathbf{E}}_i$  into another  $\underline{\mathbf{E}}_t$ . Here  $i$  stands for “incident” and  $t$  for “transmitted.” Therefore such a device can be represented as  $2 \times 2$  Jones matrices  $a_{ij}$  such that

$$\begin{bmatrix} E_{tx} \\ E_{ty} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E_{ix} \\ E_{iy} \end{bmatrix} \quad (1.49)$$

Below we list the Jones matrices for the most common polarizing optical elements:

Linear polarizer oriented along $x$ axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Linear polarizer oriented along $y$ axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Linear polarizer at $+45^\circ$ to $x$ axis	$\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	Linear polarizer at $-45^\circ$ to $x$ axis	$\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
Quarter-wave plate, fast axis along $x$	$\begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix}$	Quarter-wave plate, fast axis along $y$	$\begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}$

For passage through a stack of optical elements  $A_1$ , then  $A_2$ , then  $A_3$ , and so on up to  $A_n$ , the transmitted Jones vector would be

$$\underline{E}_t = A_n \cdot A_{n-1} \cdots A_2 \cdots A_1 \underline{E}_i \quad (1.50)$$

Since the matrices do not commute, they must be multiplied in the correct sequence.

Thus, for example, we see that a homogeneous *right-hand circular polarizer* which is made by first passing light through a  $+45^\circ$  linear polarizer followed by a quarter-wave plate with the fast axis along the  $y$  axis would be given by

$$\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ j & j \end{bmatrix} \quad (1.51)$$

Similarly, two quarter-wave plates stacked together, with both fast axes along  $y$ , yield

$$\begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (1.52)$$

Note that the resultant matrix rotates a  $\pm 45^\circ$  polarized beam by  $90^\circ$ , which makes the combination a half-wave plate.

## 1.5 REFLECTION AND REFRACTION

Consider a plane monochromatic wave of amplitude  $E_i$  incident on an interface between two media. The plane of incidence is defined as the plane containing the normal to the interface and the direction of propagation of the incident wave (see Fig. 1.6).

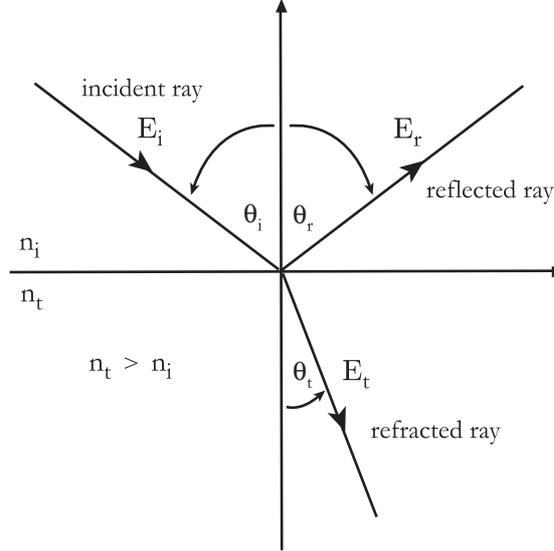


Figure 1.6: Reflection and refraction at a planar interface between two media characterized by indices of refraction  $n_i$  and  $n_t$ .

The laws of reflection are:

1. The reflected ray lies in the plane of incidence.
2. The angle of reflection is equal to the angle of incidence. That is,  $\theta_i = \theta_r$ .

The laws of refraction are:

1. For isotropic materials, the angle of refraction lies in the plane of incidence.
2. The sine of the angle of refraction bears a constant ratio to the sine of the angle of incidence. Specifically,

$$n_i \sin \theta_i = n_t \sin \theta_t \quad (1.53)$$

where  $n_i$  and  $n_t$  are the refractive indices of the media containing the incident and transmitted (refracted) waves respectively. The above equations follow from the phase matching conditions at the interface. These are

$$\mathbf{k}_i^{\text{tang}} = \mathbf{k}_r^{\text{tang}} = \mathbf{k}_t^{\text{tang}} \quad (1.54)$$

where “tang” means tangential to the interface. Waves that are polarized transverse (perpendicular) to the plane of incidence are called Transverse Electric (*TE*) waves, and waves

polarized within (parallel to) the plane of incidence are called Transverse Magnetic (*TM*) waves.

Returning to Fig. 1.6, we may represent the incident wave with the help of Eqns. 1.17 and 1.18 by

$$\mathbf{E}_i(\mathbf{r}, t) = \Re \left\{ \hat{e} \underline{E}_i(\mathbf{r}, t) e^{j(\mathbf{k}_i \cdot \mathbf{r} - \omega t)} \right\} \quad (1.55)$$

where

$$\underline{E}_i = E_i e^{j\phi_i} \quad (1.56)$$

and  $\phi_i$  is the reference phase of the incident beam. Similarly, we may write the reflected wave as

$$\mathbf{E}_r(\mathbf{r}, t) = \Re \left\{ \hat{e} \underline{E}_r(\mathbf{r}, t) e^{j(\mathbf{k}_r \cdot \mathbf{r} - \omega t)} \right\} \quad (1.57)$$

where

$$\underline{E}_r = E_r e^{j\phi_r} \quad (1.58)$$

The *complex reflection and transmission* coefficients at the interface between two dielectric media are defined by

$$\underline{r} = \underline{E}_r / \underline{E}_i = r e^{j(\phi_r - \phi_i)} = r e^{j\delta\phi_r} \quad (1.59)$$

where  $\delta\phi_r = \phi_r - \phi_i$ , and

$$\underline{t} = \underline{E}_t / \underline{E}_i = t e^{j\delta\phi_t} \quad (1.60)$$

with  $\delta\phi_t = \phi_t - \phi_i$ .

These coefficients are polarization dependent, and are given below by Fresnel's equations. The Fresnel equations can be derived from the plane wave solutions for  $\mathbf{E}(\mathbf{r})$  and  $\mathbf{H}(\mathbf{r})$  by: (1) noting that these fields satisfy the space-dependent Helmholtz Equation  $[(\nabla^2 + k^2)\underline{\mathbf{E}}(r) = 0]$ , (2) writing out  $\mathbf{k}_i$ ,  $\mathbf{k}_r$  and  $\mathbf{k}_t$  explicitly in terms of  $\theta_i$ ,  $n_i$ ,  $n_t$  and the coordinate axes, and (3) applying the appropriate boundary conditions ( $E_{tan}$ ,  $H_{tan}$ ,  $B_{norm}$  and  $D_{norm}$  are continuous) at the boundary between the two media.

The reflection coefficients for *TE* and *TM* waves are denoted by  $r_{\perp}$  and  $r_{\parallel}$ , respectively, where the subscripts  $\perp$  and  $\parallel$  refer to the polarization with respect to the plane of incidence. For the *TE* wave, the Fresnel equation is

$$r_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} = \frac{n_i \cos \theta_i - n_t \cos \theta_t}{n_i \cos \theta_i + n_t \cos \theta_t} = \frac{\cos \theta_i - \sqrt{(n_{ti}^2 - \sin^2 \theta_i)}}{\cos \theta_i + \sqrt{(n_{ti}^2 - \sin^2 \theta_i)}} \quad (1.61)$$

where  $n_{ti} = n_t/n_i$ . For the *TM* wave we have

$$r_{\parallel} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = \frac{n_t \cos \theta_i - n_i \cos \theta_t}{n_t \cos \theta_i + n_i \cos \theta_t} = \frac{n_{ti}^2 \cos \theta_i - \sqrt{(n_{ti}^2 - \sin^2 \theta_i)}}{n_{ti}^2 \cos \theta_i + \sqrt{(n_{ti}^2 - \sin^2 \theta_i)}} \quad (1.62)$$

The corresponding transmission coefficients are given by

$$\underline{t}_\perp = 1 + \underline{r}_\perp = \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_i + \theta_t)} = \frac{2n_i \cos \theta_i}{n_i \cos \theta_i + n_t \cos \theta_t} \quad (1.63)$$

and

$$\underline{t}_\parallel = \frac{1}{n_{ti}}(1 + \underline{r}_\parallel) = \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_i + \theta_t) \cos(\theta_i - \theta_t)} = \frac{2n_i \cos \theta_i}{n_t \cos \theta_i + n_i \cos \theta_t} \quad (1.64)$$

The case where the light is incident on the interface from the lower refractive index medium is called external reflection. Internal reflection corresponds to the case where the light is incident on the interface from the higher index material. If we let  $\underline{r}$  and  $\underline{t}$  represent the reflection and transmission coefficients for external reflection and  $\underline{r}'$  and  $\underline{t}'$  the reflection and transmission coefficients for internal reflection, it is clear from Fresnel's equation's that

$$\underline{r} = -\underline{r}' \quad (1.65)$$

and that in general

$$|\underline{t}| \neq |\underline{t}'| \quad (1.66)$$

The reflectivity  $R$  (also called the reflectance), is defined by

$$R = \underline{r}\underline{r}^* = |\underline{r}|^2 \quad (1.67)$$

and the transmittance,  $T$ , as shown later, is defined by

$$T = \underline{t}\underline{t}' \quad (1.68)$$

Thus, at any boundary,

$$R + T = 1 \quad (1.69)$$

as required by the conservation of energy.

### 1.5.1 External Reflection

External reflection refers to the case in which the incident wave propagates in the lower-index medium before reflection from the interface. That is, when  $n_i < n_t$ , where the subscripts  $i$  and  $t$  stand for the incident and transmitted medium respectively. Thus, Fig. 1.6 as drawn depicts the case for external reflection since  $\theta_i > \theta_t$ . In this case, all of the above coefficients ( $\underline{r}_\parallel, \underline{r}_\perp, \underline{t}_\parallel, \underline{t}_\perp$ ) are real.

At normal incidence, ( $\theta_i = 0$ ), we see from both Eqn. 1.61 and Eqn. 1.62 that

$$R_\perp = R_\parallel = \left[ \frac{n_t - n_i}{n_t + n_i} \right]^2 \quad (1.70)$$

For example, a glass-air interface, ( $n_i = 1, n_t = 1.5$ ) we find that  $R_{normal} \sim 4\%$ . From Eqn. 1.62 we see that there is a special value of  $\theta_i$  for which  $r_\parallel$  vanishes. This angle is called the *Brewster angle*,  $\theta_B$ . Thus, with  $\theta_i = \theta_B$ , we see that  $\theta_B$  satisfies

$$n_{ti}^2 \cos \theta_B = \sqrt{(n_{ti}^2 - \sin^2 \theta_B)} \quad (1.71)$$

where  $n_{ti} = n_t/n_i$ . From this equation it follows that

$$\theta_B = \tan^{-1} n_{ti} \quad (1.72)$$

For the glass-air interface,  $\theta_B = 57^\circ$ .

The pair of curves to the right in Fig. 1.7 are plots of the external reflectivity as a function of the angle of incidence for *TE* and *TM* waves reflecting off an interface bounded by materials of refractive indices 1.0 and 1.5.

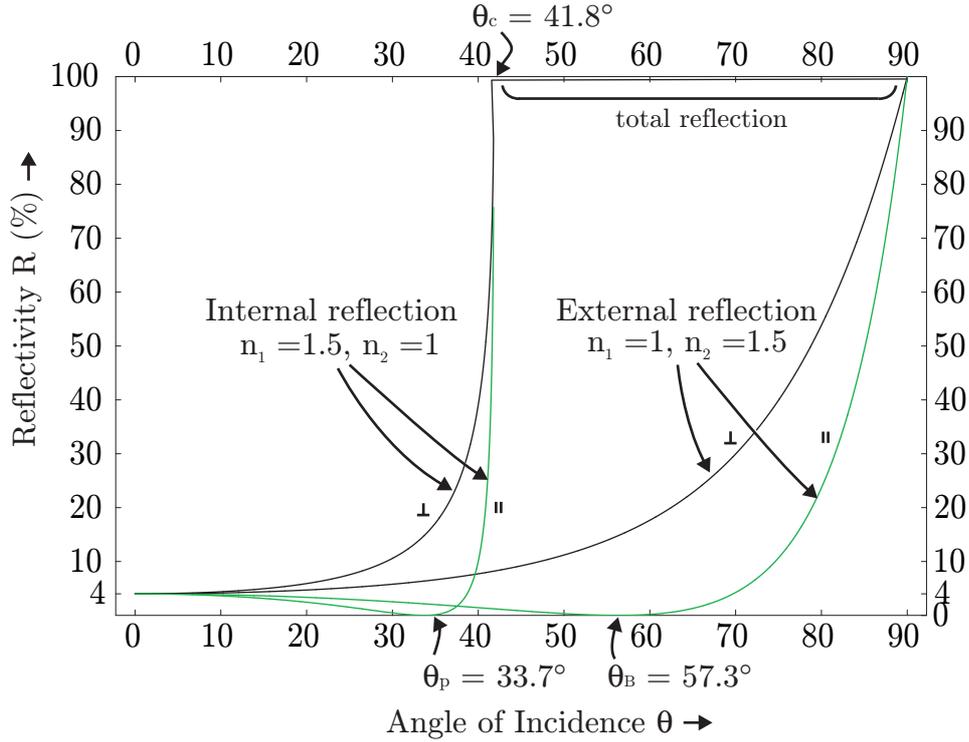


Figure 1.7: External and internal reflectivity of an interface between two dielectric media as a function of the angle of incidence

## 1.5.2 Internal Reflection

Figures 1.8(a,b) depict the case for internal reflection ( $n_i > n_t$ ). That is, the incident light is propagating from within the higher refractive index medium (incident medium) towards the lower refractive index medium (transmitted medium). In this case  $\theta_i < \theta_t$ . With  $n_i > n_t$  we see that there is also a Brewster angle for internal reflection. It is often called the polarizing angle  $\theta_p$  and is also given by Eqn. 1.72. Notice that  $(\theta_B + \theta_p) = \pi/2$ . For internal reflection, we also see that there exists an angle of incidence called the critical angle  $\theta_c$  for which the angle of refraction is  $90^\circ$ . That is

$$\theta_c = \sin^{-1}(n_{ti}) \quad (1.73)$$

For a glass-air interface,  $n_t = 1$ ,  $n_i = 1.5$ , and we find  $\theta_c = 41.8^\circ$ . Note that there is no  $\theta_c$  for external reflection.

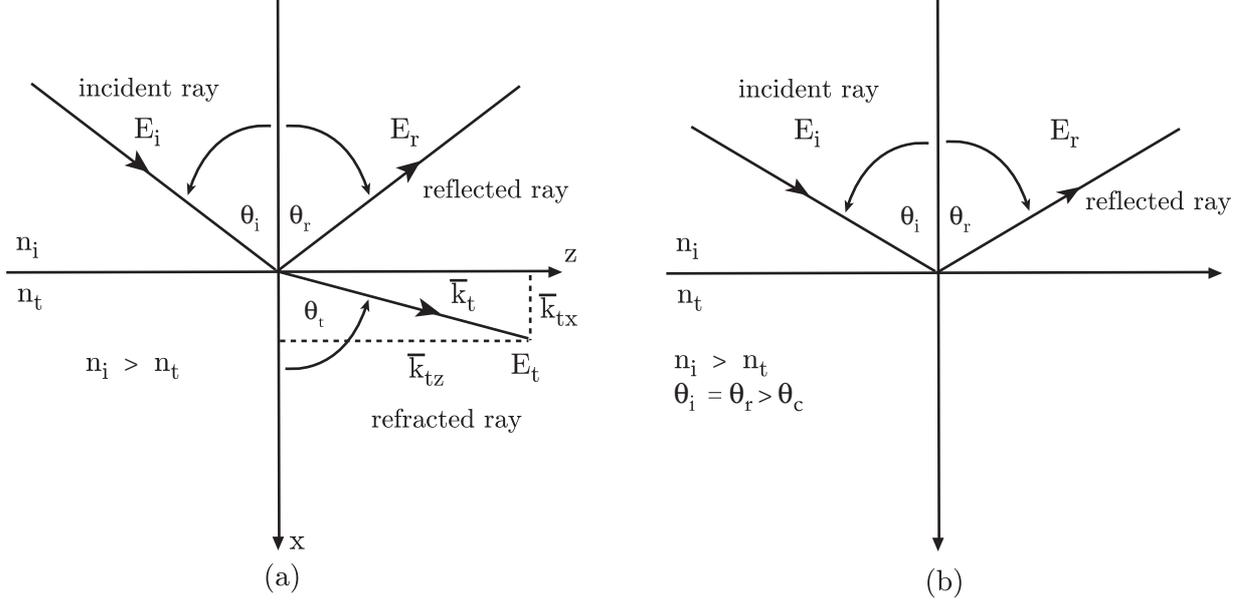


Figure 1.8: (a) Internal reflection below the critical angle; (b) Total internal reflection (above the critical angle)

When  $\theta_i$  is greater than the critical angle  $\theta_c$ , total internal reflection takes place (see Fig. 1.8(b)) and there is no transmitted propagating wave. However, an evanescent wave (non-propagating) does exist at the interface in the lower refractive index material as described below. In this case, since  $\sin \theta_i > \sin \theta_c = n_{ti}$ ,  $(n_{ti}^2 - \sin^2 \theta_i)^{1/2}$  is complex, and the reflectivity  $R$  for both polarizations is unity as implied by Eqn. 1.61 and 1.62.

The pair of curves on the left in Fig. 1.7 are plots of the internal reflectivities for  $TE$  and  $TM$  polarization from a dielectric interface between two media. The two above-mentioned important features are clearly seen:

1. The Brewster angles  $\theta_B$  and  $\theta_p$  for external and internal reflection, respectively, at which the parallel polarization component ( $TM$  wave) of the reflected light vanishes, and
2. The critical angle  $\theta_c$  above which total internal reflection takes place.

### 1.5.3 Evanescent Field

An evanescent wave is non-propagating and therefore has no energy flow even though it has an associated electric field. When  $\theta_i > \theta_c$ , as in Fig. 1.8(b), an evanescent wave exists in

the lower-index medium, but there is no power transfer into that medium normal to the interface.

To determine the penetration depth of this evanescent field into the lower-index medium, let us suppose that the “transmitted” wave can still be written in the general form

$$E_t = \Re \left\{ A e^{j(\mathbf{k}_t \cdot \mathbf{r} - \omega t)} \right\} \quad (1.74)$$

where

$$\mathbf{r} = x\hat{x} + z\hat{z} \quad (1.75)$$

and

$$\begin{aligned} \mathbf{k}_t &= k_{tx}\hat{x} + k_{tz}\hat{z} \\ &= \hat{x}kn_t \cos \theta_t + \hat{z}kn_t \sin \theta_t \end{aligned} \quad (1.76)$$

as illustrated in Fig. 1.8(b), and where  $k = 2\pi/\lambda$  is the free-space wave number. Then

$$E_t = \Re \left\{ A e^{j[kn_t(z \sin \theta_t + x \cos \theta_t) - \omega t]} \right\} \quad (1.77)$$

But for total internal reflection,  $\cos \theta_t$  is imaginary. Thus we can rewrite  $\cos \theta_t$  as,

$$\cos \theta_t = \left[ 1 - \left( \frac{n_i}{n_t} \right)^2 \sin^2 \theta_i \right]^{1/2} = \pm j\gamma \quad (1.78)$$

where

$$\gamma = \left( \frac{n_i}{n_t} \right) \left[ \sin^2 \theta_i - \left( \frac{n_t}{n_i} \right)^2 \right]^{1/2} \quad (1.79)$$

Thus,

$$\begin{aligned} E_t &= \Re \left\{ A e^{j \left\{ kn_t \left[ z \left( \frac{n_i}{n_t} \right) \sin \theta_i \pm j\gamma x \right] - \omega t \right\}} \right\} \\ &= \Re \left\{ A e^{\pm j\gamma kn_t x} e^{j[zkn_i \sin \theta_i - \omega t]} \right\} \end{aligned} \quad (1.80)$$

The plus sign in the exponent implies amplification of the wave with distance away from the interface; this solution cannot be allowed in this case, as it violates energy conservation. The negative sign implies that the magnitude of the evanescent wave amplitude decays exponentially with distance away from the interface. That is,  $|E_t|$  is of the form

$$|E_t| = A e^{-\beta x} \quad (1.81)$$

where

$$\begin{aligned} \beta &= (2\pi/\lambda)n_t\gamma \\ &= \frac{2\pi}{\lambda} n_i (\sin^2 \theta_i - n_t^2/n_i^2)^{1/2} \end{aligned} \quad (1.82)$$

The *penetration depth*  $d_p$  of the evanescent field into the less-dense medium is defined by  $d_p = 1/\beta$ . That is, the penetration depth is defined as the distance away from the interfaces over which the evanescent field falls to  $\frac{1}{e}$  of its value at the interface. Using our example for the glass-air interface above, we find that for  $\theta_i = 60^\circ$ ,  $d_p = 0.19\lambda$ ; and, at a distance of one wavelength away from the interface, the evanescent field has decayed by a factor of  $e^{-5.21} (= 5.5 \times 10^{-3})$  of its value at the interface.

## 1.6 PHASE CHANGES ON REFLECTION AND TRANSMISSION

For external reflection,  $\theta_i > \theta_t$ , and internal reflection,  $\theta_i < \theta_t$ , the Fresnel equations for angles of incidence below the Brewster angles, lead to the table below for the phase changes for the reflected and transmitted beams:

External Reflection	Internal Reflection
$\delta\phi_{r\perp} = \phi_{r\perp} - \phi_i = \pi$	$\delta\phi'_{r\perp} = \phi'_{r\perp} - \phi_i = 0$
$\delta\phi_{r\parallel} = \phi_{r\parallel} - \phi_i = 0$	$\delta\phi'_{r\parallel} = \phi'_{r\parallel} - \phi_i = \pi$
$\delta\phi_{t\parallel} = \phi_{t\parallel} - \phi_i = 0$	$\delta\phi'_{t\parallel} = \phi'_{t\parallel} - \phi_i = 0$
$\delta\phi_{t\perp} = \phi_{t\perp} - \phi_i = 0$	$\delta\phi'_{t\perp} = \phi'_{t\perp} - \phi_i = 0$

Thus, we see that irrespective of the polarization of the wave,

$$|\delta\phi_r - \delta\phi'_r| = \pi, \quad (1.83)$$

$$\delta\phi_{t\perp} = \delta\phi'_{t\perp} = 0 \quad (1.84)$$

and that

$$\delta\phi_{t\parallel} = \delta\phi'_{t\parallel} = 0 \quad (1.85)$$

To compute the phase changes  $\delta\phi_r$  on reflection from an interface as a function of the angle of incidence, we rewrite Fresnel's equations in the form

$$\underline{r}_\perp = r_\perp e^{j\delta\phi_\perp} \quad \text{and} \quad \underline{r}_\parallel = r_\parallel e^{j\delta\phi_\parallel} \quad (1.86)$$

and explicitly calculate the phase changes  $\delta\phi_\perp$  and  $\delta\phi_\parallel$ . For external reflection, the phase changes  $\delta\phi_\perp$  and  $\delta\phi_\parallel$  are as plotted in Fig. 1.9(a,b) as a function of the angle of incidence. In the case of total internal reflection, because  $[(n_t/n_i)^2 - \sin^2\theta_i]^{1/2}$  becomes imaginary, Eqn. 1.61, for example, can be rewritten in the form

$$\underline{r}_\perp = \frac{\underline{E}_{r\perp}}{\underline{E}_{i\perp}} = \frac{A + jB}{A - jB} = \frac{Ce^{j\phi}}{Ce^{-j\phi}} = e^{j\delta\phi'_{r\perp}} \quad (1.87)$$

where  $A = \cos\theta_i$ ,  $B = [\sin^2\theta_i - (n_t/n_i)^2]^{1/2}$ , and  $\delta\phi'_{r\perp}$  is the *TE* wave phase shift for internal reflection. It follows that

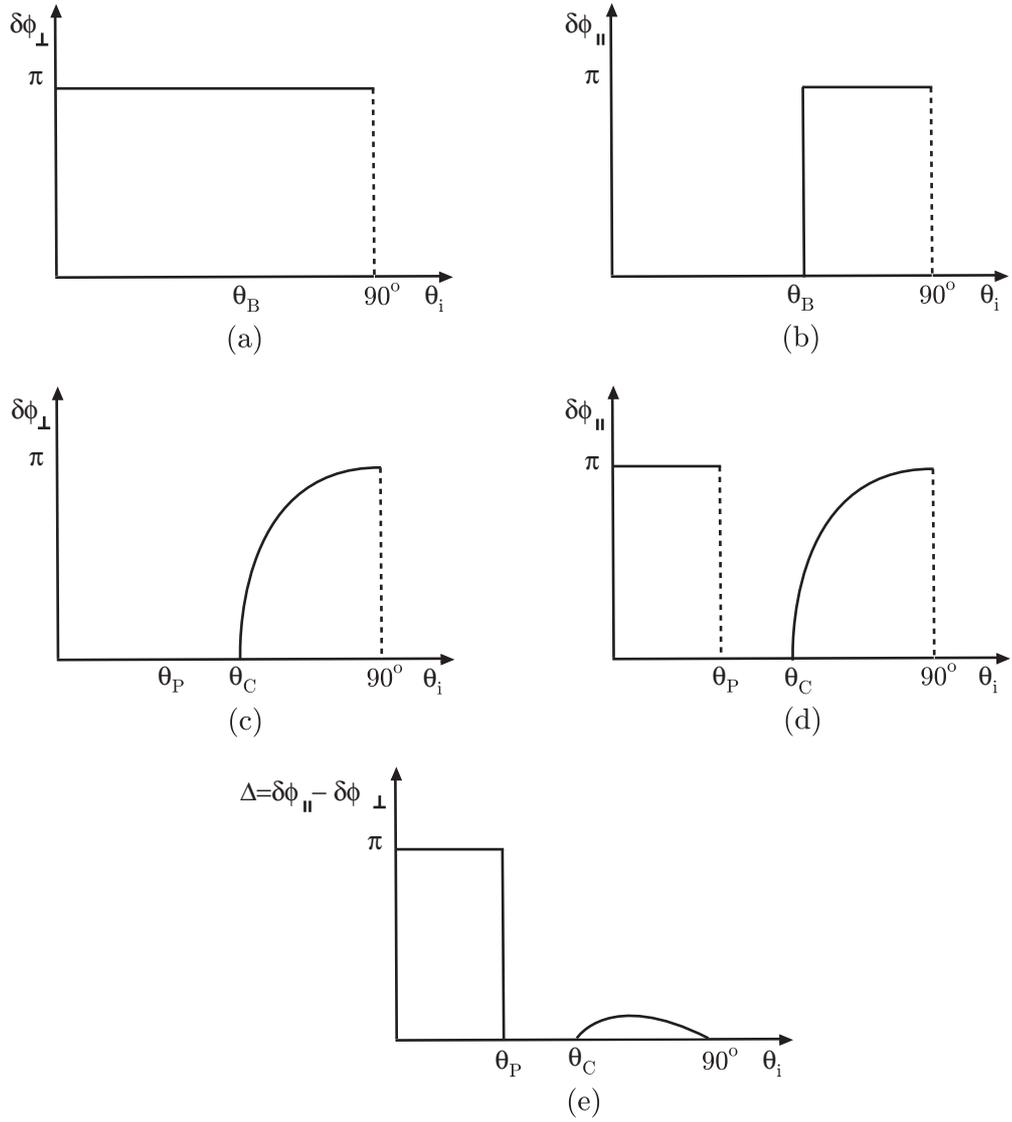


Figure 1.9: (a,b) Phase changes for external reflection from an interface between two dielectric media; (c,d) Phase changes for internal reflection; (e) Differential phase change for internal reflection

$$\tan \frac{\delta\phi'_{\perp}}{2} = \frac{[\sin^2 \theta_i - (n_t/n_i)^2]^{1/2}}{\cos \theta_i} \quad (1.88)$$

Similarly, it can be shown that for the *TM* wave under internal reflection conditions

$$\underline{r}_{\parallel} = \frac{\underline{E}_{r\parallel}}{\underline{E}_{i\parallel}} = e^{j\delta\phi_{\parallel}} \quad (1.89)$$

where

$$\tan \delta\phi_{\parallel}/2 = \left(\frac{n_i}{n_t}\right)^2 \tan \delta\phi_{\perp}/2$$

$$= \frac{[\sin^2 \theta_i - (n_t/n_i)^2]^{1/2}}{(n_t/n_i)^2 \cos \theta_i} \quad (1.90)$$

We note that  $E_r$  leads  $E_i$  in phase. From Eqn. 1.88 and 1.90 it follows that the relative phase difference for total internal reflection,  $\Delta = \delta\phi_{\parallel} - \delta\phi_{\perp}$  is given by,

$$\tan \frac{\Delta}{2} = \frac{\cos \theta_i [\sin^2 \theta_i - (n_t/n_i)^2]^{1/2}}{\sin^2 \theta_i} \quad (1.91)$$

The phase changes for internal reflection are plotted below in Fig. 1.9(c,d,e). As an example, consider again the glass-air interface. In this case  $\theta_c = \sin^{-1}(1/1.5) = 41.8^\circ$ . If we let  $\theta_i = 60^\circ$ , then we find that for total internal reflection,  $\tan \delta\phi_{\parallel} = 1.106$  which implies  $\delta\phi_{\parallel} = 0.27\pi$  radians, and  $\delta\phi_{\perp} = 0.38\pi$  radians.

## 1.7 THE STOKES TREATMENT OF REFLECTION AND REFRACTION

The stokes treatment of reflection and transmission at a boundary is an elegant and novel way of establishing the relationships between the reflection and transmission coefficients as well as the phase changes on reflection and transmission. Consider a wave of amplitude  $E_i$  incident on the interface between two dielectric media of refractive indices  $n_1$  and  $n_2$  ( $n_2 > n_1$ ) as shown in Fig. 1.10(a). Let  $\underline{r}$  and  $\underline{t}$  be the reflection and transmission coefficients, respectively, for a wave incident from the less dense medium on the more dense medium (external reflection), and let  $\underline{r}'$  and  $\underline{t}'$  be the amplitude reflection and transmission coefficients, respectively, for a wave incident from the more dense onto the less dense medium (internal reflection). Based on Fermat's principle (the principle of reversibility) time-reversal invariance holds, and so does the situation shown in Fig. 1.10(b), where the original reflected and transmitted ray directions are reversed to become incident beams. In effect, this situation must also be physically possible if there is no absorption at the interface.

In Fig. 1.10(b), we see that the incident wave from above has amplitude  $\underline{r}E_i$  and the wave incident from below has amplitude  $\underline{t}E_i$ . A portion of each of these waves is both reflected and transmitted at the interface as shown. The reflected and transmitted portions of  $\underline{r}E_i$  are  $r^2E_i$  and  $\underline{r}\underline{t}E_i$ , and similarly the reflected and transmitted portions of  $\underline{t}E_i$  are  $\underline{r}'\underline{t}E_i$  and  $\underline{t}'E_i$ . Since the time-reversed configuration of Fig. 1.10(b) must be identical with that in Fig. 1.10(a), it follows that

$$\underline{r}\underline{t} + \underline{t}\underline{r}' = 0 \quad (1.92)$$

or

$$\underline{r}' = -\underline{r} \quad (1.93)$$

and

$$\underline{t}\underline{t}' + |\underline{r}|^2 = 1 \quad (1.94)$$

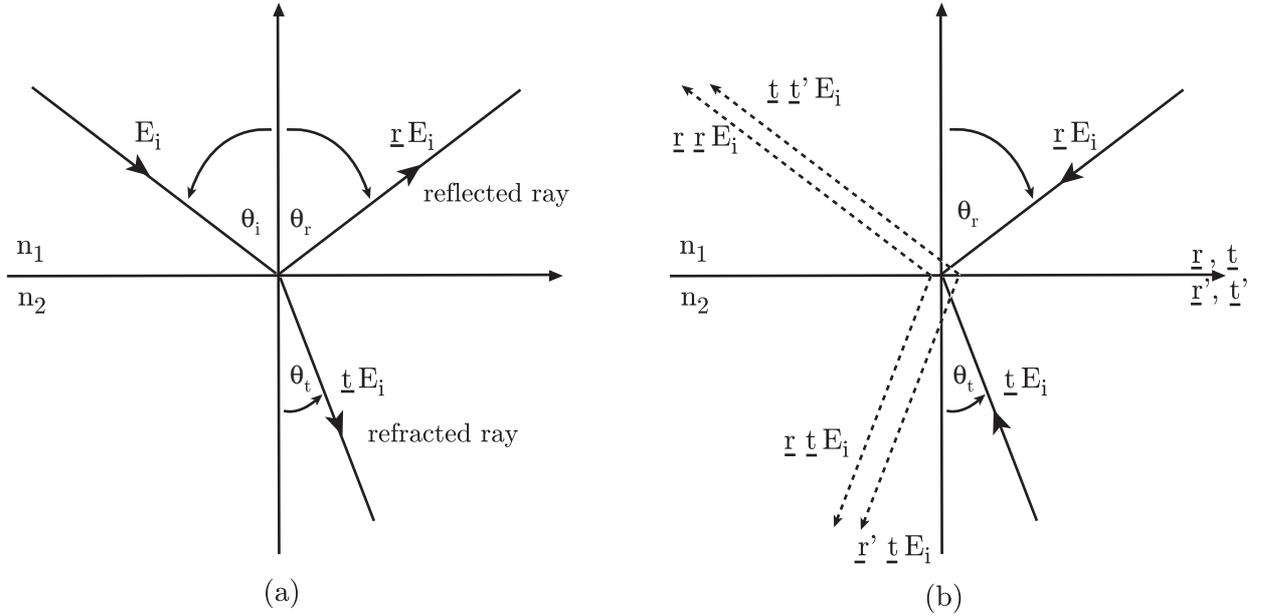


Figure 1.10: Reflection and refraction via the Stokes treatment; (a) normal case; (b) time reversed case

since  $\underline{r}$  is real. The latter two equations are called the Stokes relations. The first of the Stokes relations says that there is a phase difference of  $\pi$  radians between waves internally and externally reflected (as we have seen earlier). That is,

$$|\delta\phi_r - \delta\phi_r'| = \pi \quad (1.95)$$

irrespective of the polarization of the light. The second is just a statement of the conservation of energy since it implies that

$$T + R = 1 \quad (1.96)$$

where  $R$  and  $T$  are the reflectivity and transmissivity of the interface, with  $T$  defined by  $T = tt'$ . Finally, by combining the two Stokes relations, it also follows that

$$\underline{tt'} - \underline{rr'} = 1 \quad (1.97)$$

## 1.8 Interaction of Light with Dielectric Matter: Classical Electron Oscillator Model

### Atomic Dipole Model of Materials

One of the simplest ways of modelling a dielectric material is to think of the material as a collection of atoms where each atom is modelled as an electronic dipole. Let us assume, then, that there are  $N$  such dipoles per unit volume of the material. When an electromagnetic wave is incident on the material, the electric field of the wave drives the electrons into oscillation

at the frequency of the wave. However, for simplicity, let us assume that there is only one responding electron per atom. This displacement of the electron from its normal position relative to its positively charged and more massive nucleus leads to an induced time-varying dipole moment  $\mathbf{p}(t)$ , where

$$\mathbf{p}(t) = q\mathbf{x}(t) \quad (1.98)$$

where  $q$  is the electron charge and  $\mathbf{x}(t)$  is the time varying displacement of the electron produced by the electromagnetic wave. Thus, the field-induced polarization  $\mathbf{P}(t)$  within the material (defined as the dipole moment per unit volume) is given by

$$\mathbf{P}(t) = N\mathbf{p}(t) = Nq\mathbf{x}(t) \quad (1.99)$$

where  $N$  is the number of atoms per unit volume.

If this collision between the photon and the atom is elastic, the dipoles will reradiate an electromagnetic wave at the same frequency as the driving wave, although in general there may be a phase shift between the incident and the radiated wave. The collection of oscillating dipoles can be thought of as forming an antenna array. When the material is totally transparent, the array re-radiates in such a way that there are perfectly phased for constructive interference in the forward direction, and the electromagnetic waves thus appear to propagate without a change of direction or a loss of power.

It can be shown that the far-field radiation field,  $\underline{E}_{ff}$ , from a single z-directed oscillating electric dipole (see Figure 1.11), expressed in spherical coordinates, is of the form,

$$\underline{E}_{ff}(\mathbf{r}) = \hat{\theta} \frac{j\eta_0 k I d}{4\pi r} e^{jkr} \sin \theta \quad (1.100)$$

where  $I$  is the effective dipole current strength,  $d$  is the dipole length,  $k = 2\pi/\lambda$  is the wave vector magnitude and  $\eta_0$  is the impedance of free space. Thus, as shown in Figure 1.11, there is no power radiated along the axis of the dipole.

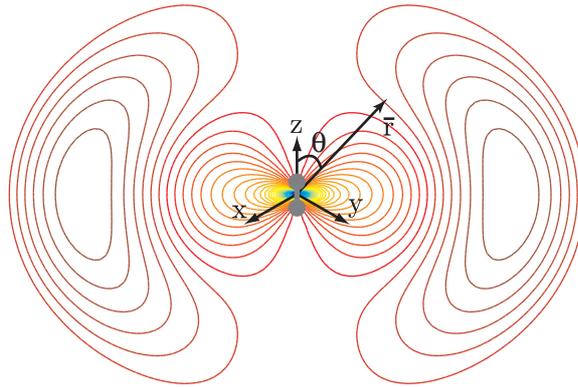


Figure 1.11: Electric field radiation Pattern from an elementary dipole

## Reflection from Interfaces Reconsidered

For the lossless case we can use the dipole model to explain the reflection of light from dielectric surfaces. Let a TM wave be incident at an angle  $\theta$  with respect to the normal on the interface as illustrated in Fig. 1.12. The field due to this wave drives dipoles near the surface of the material and these dipoles re-radiate the reflected and transmitted waves. From our earlier discussion it can be readily shown that at the Brewster angle, the angle between the refracted ray and the ‘would-be’ reflected ray is  $90^\circ$ . So, at the Brewster angle, the dipoles in the medium are driven by the refracted beam so that their axes are parallel to the reflected beam, but since a dipole does not radiate along its axis, there is no reflected wave in this case. This simple model is also useful for explaining why light scattered from the overhead sky is polarized at sunrise and sunset.

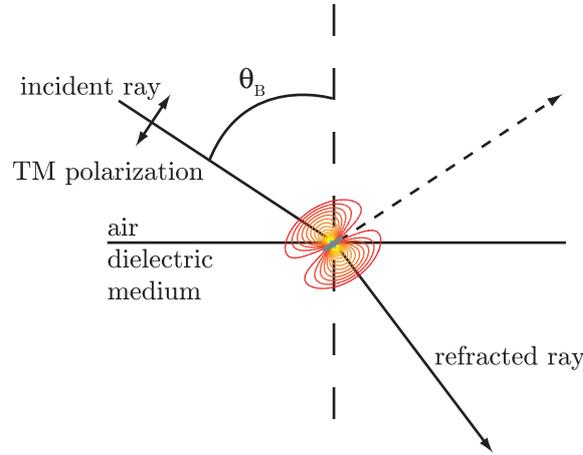


Figure 1.12: Dipole model for reflection of light from a dielectric interface at the Brewster angle

## Derivation of the Refractive Index of the Material

Consider now a typical atom deep within material that is illuminated by an optical field. Let us again assume the atom has a single electron. Let us assume that the optical field  $E$  is a plane monochromatic wave of frequency  $\omega$ , polarized in the  $\hat{x}$  direction and propagating in the  $z$  direction. That is

$$\mathbf{E}(z, t) = \Re \left\{ \hat{x} |\underline{E}| e^{j(kz - \omega t)} \right\} \quad (1.101)$$

In this classical description of the interaction of light with matter, it is assumed that the electric field causes the electrons of the atoms to oscillate at frequency  $\omega$ , giving rise to oscillating dipoles. These re-radiate at the same frequency as the driving field but not necessarily with the same phase.

Generally, the atom is not isolated. In a solid, for example, it is bound in a lattice, and damping forces will be present. Such a material is absorbing (lossy), and the absorption is generally frequency dependent. Thus power will be removed from the wave as it

propagates through the material. This loss mechanism leads to anomalous dispersion in the neighborhood of the resonant absorption frequency as we shall see below.

The classical electron oscillator model, or the Lorentz model as it is sometimes called, assumes that the electron oscillates as a harmonic oscillator about its equilibrium position. Let  $m$  be the effective mass of the electron,  $q$  the electronic charge,  $\gamma$  the velocity damping constant and  $\omega_0$  is the natural frequency of the atomic system. Then, assuming the electron is localized at  $z = 0$  in the absence of the optical field, the net force acting on the electron is

$$F(x, t) = q|\underline{E}|\Re\{e^{-j\omega t}\} - kx - 2\gamma\frac{dx}{dt} \quad (1.102)$$

where the first term is that owing to the optical field, the second is the restoring force due to electrostatic attraction towards the nucleus, and the third term is the velocity damping force. Since we have modelled the electrostatic restoring force with Hooke's law using a spring constant  $k$ , we may write  $k = m\omega_0^2$  where  $\omega_0$  is the natural frequency of oscillation of the system. Thus, using  $F = m\frac{d^2x}{dt^2}$  we arrive at the equation of motion for the electron with the atom located at  $z = 0$ .

$$m\left[\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2x\right] = q|\underline{E}|\Re\{e^{-j\omega t}\} \quad (1.103)$$

The transient (homogeneous) solution  $x_h$  to this differential equation is

$$x_h(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t} \quad (1.104)$$

where

$$\alpha_1 = -\gamma + j(\omega_0^2 - \gamma^2)^{1/2} \quad (1.105)$$

and

$$\alpha_2 = -\gamma - j(\omega_0^2 - \gamma^2)^{1/2} \quad (1.106)$$

and where we have assumed that  $\omega_0 \gg \gamma$ . Because  $\gamma > 0$ , this solution dies exponentially. The more important steady-state or time-harmonic (particular) solution  $x_p(t)$  is of the form

$$x_p(t) = \underline{C}\Re\{e^{-j\omega t}\} \quad (1.107)$$

where  $\underline{C}$  is a complex constant that includes the phase shift between the driving electric field,  $\underline{E}$ , of the light and the resulting displacement is  $x_p(t)$ . Substitution of 1.107 into 1.103 and solving for  $\underline{C}$  leads to

$$x_p(t) = \frac{q|\underline{E}|\Re\{e^{-j\omega t}\}}{m(\omega_0^2 - 2j\gamma\omega - \omega^2)} \quad (1.108)$$

If there are  $N$  active electrons per unit volume in the material, then the polarization density  $P_x$ , (i.e., the steady-state dipole moment per unit volume) is

$$P_x(t) = Nqx_p(t) \quad (1.109)$$

$$= \frac{Nq^2|\underline{E}|\Re\{e^{-j\omega t}\}}{m(\omega_0^2 - 2j\gamma\omega - \omega^2)} \quad (1.110)$$

Assuming that  $|\underline{E}|$  is sufficiently small,  $\underline{P}$  will be linear with  $\underline{E}$ ; that is,

$$\underline{P} = \alpha \underline{E} = \epsilon_0 \underline{\chi} \underline{E} \quad (1.111)$$

where  $\alpha$  is the polarizability, and  $\underline{\chi}$  is the electric susceptibility. Thus, it follows that the susceptibility tensor in our example is given by

$$\underline{\chi}(\omega) = \frac{Nq^2}{m\epsilon_0} \left\{ \frac{1}{(\omega_0^2 - 2j\gamma\omega - \omega^2)} \right\} \quad (1.112)$$

As we have seen before, the electric displacement  $\underline{D}$  is given by

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P} \quad (1.113)$$

but

$$\underline{D} = \epsilon \underline{E} \quad (1.114)$$

Thus, using Eqn. 1.11, we see that the dielectric constant,  $\epsilon/\epsilon_0$ , of the cloud of atoms is given by

$$\epsilon(\omega)/\epsilon_0 = 1 + \underline{\chi}(\omega) \quad (1.115)$$

$$= 1 + \frac{Nq^2}{m\epsilon_0} \left\{ \frac{1}{\omega_0^2 - 2j\gamma\omega - \omega^2} \right\} \quad (1.116)$$

We can separate  $\epsilon(\omega)/\epsilon_0$  into its real and imaginary parts and plot the real and imaginary parts of the dielectric constant as a function of frequency. The results are

$$\Re \left\{ \frac{\epsilon(\omega)}{\epsilon_0} \right\} = 1 + \frac{Nq^2}{m\epsilon_0} \left\{ \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2} \right\} \quad (1.117)$$

$$\Im \left\{ \frac{\epsilon(\omega)}{\epsilon_0} \right\} = \frac{Nq^2}{m\epsilon_0} \left\{ \frac{2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2} \right\} \quad (1.118)$$

. The corresponding anomalous dispersion and absorption curves shown in Fig. 1.13

For non-magnetic media  $\epsilon(\omega)/\epsilon_0 \simeq \underline{n}^2(\omega)$ , where  $\underline{n}(\omega)$  is the complex index of refraction, we can write  $\underline{n}(\omega)$  in terms of its real and imaginary parts as follows:

$$\underline{n}(\omega) = n'(\omega) + jn''(\omega) \quad (1.119)$$

A great deal of the microscopic properties of matter are manifest in the complex index of refraction. It should be realized that  $n'$  and  $n''$  are not independent of each other. They are related by the dispersion relations

$$\Re\{\epsilon/\epsilon_0\} = n'^2(\omega) - n''^2(\omega) \quad (1.120)$$

and

$$\Im\{\epsilon/\epsilon_0\} = 2n'(\omega)n''(\omega) \quad (1.121)$$

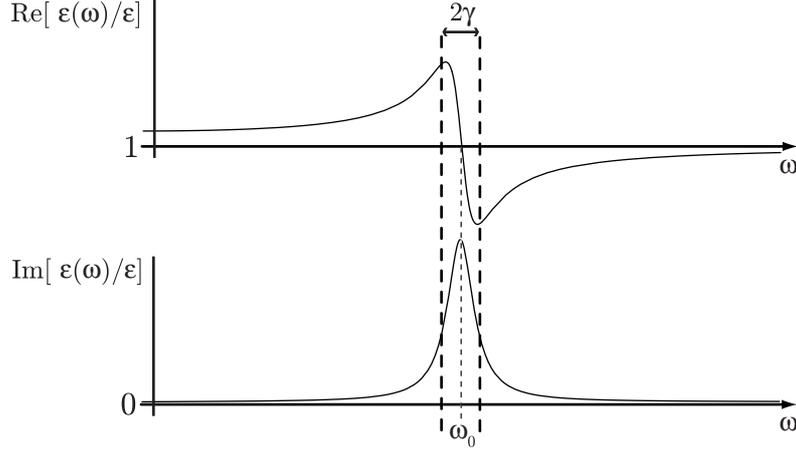


Figure 1.13: Plots of the real and imaginary parts of the dielectric constant in the neighborhood of a natural frequency (absorption band region)

Also, since

$$\underline{\epsilon}/\epsilon_0 = 1 + \underline{\chi} \quad (1.122)$$

and

$$\underline{\chi} = \chi' + j\chi'' \quad (1.123)$$

it follows that

$$\chi'' = 2n'(\omega)n''(\omega) \quad (1.124)$$

or

$$n''(\omega) = \frac{\chi''(\omega)}{2n'(\omega)} \quad (1.125)$$

Furthermore, causality requires that  $\Re\{\epsilon/\epsilon_0\}$  and  $\Im\{\epsilon/\epsilon_0\}$  satisfy the Kramers-Kronig relations:

$$[n'^2(\omega) - n''^2(\omega)] = \frac{2}{\pi} \int_0^\infty \frac{\omega' 2n'(\omega')n''(\omega')}{\omega'^2 - \omega^2} d\omega' + \text{const.} \quad (1.126)$$

and

$$2n'(\omega)n''(\omega) = -\frac{2\omega}{\pi} \int_0^\infty \frac{n'^2(\omega') - n''^2(\omega')}{\omega'^2 - \omega^2} d\omega' \quad (1.127)$$

## Plasma

Returning to Eqn. 1.117, we see that in the limit when  $\gamma \rightarrow 0$  (no damping) and  $\omega_0 \rightarrow 0$  (electron and its nucleus become decoupled, as in a plasma) and we find that

$$\Re \left\{ \frac{\epsilon(\omega)}{\epsilon_0} \right\} = \left( 1 - \frac{Nq^2}{\epsilon_0 m \omega^2} \right) \quad (1.128)$$

$$= \left( 1 - \frac{\omega_p^2}{\omega^2} \right) \quad (1.129)$$

where the *plasma frequency*,  $\omega_p$ , is given by

$$\omega_p^2 = Nq^2/m\epsilon_0 \quad (1.130)$$

The plasma frequency is the natural collective oscillation frequency of the “sea” of electrons.

The refractive index,  $n(\omega)$  of the plasma is given by

$$n(\omega) = \left( 1 - \frac{\omega_p^2}{\omega^2} \right)^{1/2} \quad (1.131)$$

Note that for  $\omega > \omega_p$ ,  $n > 0$  and we have normal propagation even though the the phase velocity exceeds the speed of light. On the other hand for  $\omega < \omega_p$ ,  $n$  is imaginary, and we have evanescence. That is, the amplitude of the wave decays exponentially as it enters the plasma, and the wave is strongly reflected off the plasma.

### Absorption Coefficient

The imaginary part of the refractive index can be measured by means of absorption spectroscopy. If a plane wave of frequency  $\omega$  is incident on an absorbing material characterized by a complex refractive index

$$\underline{n}(\omega) = n'(\omega) + jn''(\omega) \quad (1.132)$$

its propagation vector  $\underline{k}(\omega) = n(\omega)\omega/c$  will similarly be complex-valued. That is,

$$\underline{k}(\omega) = \frac{n'(\omega)\omega}{c} + j\frac{n''(\omega)\omega}{c} \quad (1.133)$$

Thus, in the material, the plane wave will be given by

$$E(z, t) = \Re \left\{ |E_0| e^{j(n'\omega z/c - \omega t)} e^{-n''\omega z/c} \right\} \quad (1.134)$$

Note that the velocity of the wave in the material is reduced by  $c/n'$  and the wave is exponentially damped as it propagates. On squaring the above expression to arrive at the intensity, we obtain

$$I(z) = I_0 e^{-\frac{2n''\omega z}{c}} \quad (1.135)$$

This is of the familiar form,

$$I(z) = I_0 e^{-a(\omega)z} \quad (1.136)$$

where  $a(\omega)$  is the absorption coefficient given by

$$a(\omega) = \frac{2\omega n''(\omega)}{c} = \frac{4\pi n''(\omega)}{\lambda} \quad (1.137)$$

$$= \frac{\omega \chi''(\omega)}{cn'(\omega)} \quad (1.138)$$

by virtue of Eqn. 1.125.

Note also that if somehow we can make the imaginary part of the susceptibility be negative, then the system will amplify the light rather than absorb it. Such is the case for the active material within a laser cavity where external energy is pumped into the material.

The normalized absorption or emission profile  $g(\omega)$  of a material is called the line-shape. It is defined by

$$g(\omega) = \pm \frac{a(\omega)}{\int_{-\infty}^{\infty} a(\omega) d\omega} \quad (1.139)$$

It can be shown that the lineshape  $g(\omega)$  implied by Eqns. 1.138 and 1.118 is Lorentzian. That is,  $g(\nu)$  is of the form

$$g(\nu) = \frac{\Delta\nu/2\pi}{(\nu - \nu_0)^2 + (\Delta\nu/2)^2} \quad (1.140)$$

where  $\nu = \omega/2\pi$ ,  $\Delta\nu = \gamma/\pi$  and  $\nu_0$  is the center frequency of the line.

In absorption spectroscopy, one measures the fractional transmission  $I_t/I_o$  as a function of wavelength for a fixed sample path length. From this measurement, the lineshape  $g(\omega)$ , the line width  $\Delta\omega$ , the absorption coefficient  $a(\omega)$  and the imaginary part of the refractive index can be obtained. Additionally, one may measure the reflectivity  $R$  of the sample at, say, normal incidence to get a second relation between  $n'$  and  $n''$ . In this case, we find

$$R = \frac{|1 - n|^2}{|1 + n|^2} = \frac{(n' - 1)^2 + n''^2}{(n' + 1)^2 - n''^2} \quad (1.141)$$

and thus both  $n'$  and  $n''$  can be obtained experimentally.