18 Kalman-Yakubovich-Popov Lemma

Kalman-Yakubovich-Popov (KYP) Lemma (also frequently called "positive real lemma") is a major result of the modern linear system theory. It is a collection of statements related to existence and properties of quadratic storage functions for LTI state space models and quadratic supply rates. The KYP Lemma is used in the derivation of H2 and H-Infinity optimal controllers, Hankel optimal reduced models, in the analysis of robustness of LTI systems, and also in convertion between frequency domain and time domain constraints.

The KYP Lemma is associated with several statements, each concerning an LTI state space model:

$$\dot{x}(t) = Ax(t) + Bw(t) \tag{18.1}$$

in continuous time, or

$$x(t+1) = Ax(t) + Bw(t) \tag{18.2}$$

in discrete time, where A, B are real matrices of dimensions n-by-n and n-by-m respectively, and a quadratic supply rate

$$\sigma: \ \mathbb{C}^n \times \mathbb{C}^m \mapsto \mathbb{R}: \quad \sigma(x,w) = \begin{bmatrix} x \\ w \end{bmatrix}' \Sigma \begin{bmatrix} x \\ w \end{bmatrix}, \tag{18.3}$$

where $\Sigma = \Sigma'$ is a given real symmetric matrix. Recall that a function $V: \ \mathbb{R}^n \mapsto \mathbb{R}$ is called a *storage function* for system (18.1) with supply rate (18.3) if

$$\int_{t_0}^{t_1} \sigma(x(t), w(t))dt \geq V(x(t_1)) - V(x(t_0)) \tag{18.4}$$

for every $t_1 \geq t_0 \geq 0$ and every solution $x = x(t)$, $w = w(t)$ of (18.1) (the integral is replaced by the corresponding sum in the DT case).

When V is *differentiable* at every point of $\mathbb{R}^n$, condition (18.4) can be re-written in the differential form

$$\sigma(x,w) - \dot{V}(x)(Ax + Bw) \geq 0 \ \ \forall\, x \in \mathbb{R}^n, \ w \in \mathbb{R}^m. \tag{18.5}$$

A similar DT condition (which does not require differentiability of V) is

$$\sigma(x,w) + V(x) - V(Ax + Bw) \geq 0 \ \ \forall\, x \in \mathbb{R}^n, \ w \in \mathbb{R}^m. \tag{18.6}$$

The KYP Lemma, being a generalization of the classical Lyapunov theorems, addresses existence and basic properties of *quadratic* storage functions for LTI state space models

with quadratic supply rates. When $V(x) = x'Px$ is a quadratic form, (18.5) is equivalent to positive semidefiniteness of a quadratic form:

$$\sigma(x, w) - 2x'P(Ax + Bw) \geq 0, \tag{18.7}$$

which is the same as positive semidefiniteness a symmetric matrix:

$$\Sigma + LPM + M'PL' \geq 0, \quad \text{where} \quad L = \begin{bmatrix} I \\ 0 \end{bmatrix}, \; M = \begin{bmatrix} A & B \end{bmatrix}.$$

Similarly, (18.6) is equivalent to

$$\sigma(x, w) + x'Px - (Ax + Bw)'P(Ax + Bw) \geq 0, \tag{18.8}$$

i.e.

$$\Sigma + LPL' - M'PM \geq 0.$$

18.1 KYP Lemma in Time Domain

The "time domain" statements of the KYP Lemma claim that *existence* of a quadratic storage function with a given quadratic supply rate is equivalent to certain other time domain properties of the state space model.

Definition 18.1 The pair of real matrices (A, B) in (18.1) is called (continuous time) stabilizable if there exists a real matrix F such that $A + BF$ is a Hurwitz matrix, i.e. all eigenvalues of $A + BF$ have negative real part. Similarly, the pair of matrices (A, B) in (18.1) is called (discrete time) stabilizable if there exists a real matrix F such that $A + BF$ is a Schur matrix, i.e. all eigenvalues of $A + BF$ have absolute value less than 1.

If the pair (A, B) is continuous time stabilizable then for every $x_0 \in \mathbb{R}^n$ the set

$$\mathcal{M}_{CT}(x_0) \stackrel{\text{def}}{=} \{(x, w) \in \mathcal{L}_n \times \mathcal{L}_m : \; \dot{x}(t) = Ax(t) + Bw(t), \; x(0) = x_0, \; \lim_{t \to +\infty} x(t) = 0\}$$

is not empty: at least, it contains the solution of

$$\dot{x}(t) = (A + BF)x(t), \quad x(0) = x_0, \quad w(t) = Fx(t).$$

Similarly, if the pair (A, B) is discrete time stabilizable then for every $x_0 \in \mathbb{R}^n$ the set

$$\mathcal{M}_{DT}(x_0) \stackrel{\text{def}}{=} \{(x, w) \in \ell_n \times \ell_m : \; x(t+1) = Ax(t) + Bw(t), \; x(0) = x_0, \; \lim_{t \to +\infty} x(t) = 0\}$$

is not empty.

Theorem 18.1 *Assume that the pair (A, B) is continuous time stabilizable. Then the following conditions are equivalent:*

(a) system (18.1) has a storage function $V : \mathbb{R}^n \mapsto \mathbb{R}$, continuous at the origin, with quadratic supply rate (18.3);

(b) system (18.1) has a quadratic storage function $V(x) = x'Px$ with quadratic supply rate (18.3);

(c) there exist $r > 0$ and $c \in \mathbb{R}$ such that

$$\int_{t_0}^{t_1} \sigma(x(t), w(t))dt > c \ \ \forall\ t_1 \geq t_0 \geq 0$$

whenever signals x, w satisfy (18.1) with $|x(t_0)| < r$ and $|x(t_1)| < r$.

If conditions (a)-(c) are satisfied then the limit

$$\Phi(x(\cdot), u(\cdot)) = \lim_{T \to +\infty} \int_0^T \sigma(x(t), w(t))dt$$

exists for all $(x, u) \in \mathcal{M}_{CT}(x_0)$, and the infimum of Φ over $\mathcal{M}_{CT}(x_0)$ equals $-x_0'Px_0$ for some matrix $P_0 = P_0'$, which satisfies (18.7), and is also such that $P_0 \leq P$ for every $P = P'$ satisfying (18.7). Moreover, for the minimal solution $P = P_0$, the quadratic form on the left side of the inequality in (18.7) has rank not larger than m.

The following statement is a discrete time analog of Theorem 18.1.

Theorem 18.2 *Assume that the pair (A, B) is discrete time stabilizable. Then the following conditions are equivalent:*

(a) system (18.2) has a storage function $V : \mathbb{R}^n \mapsto \mathbb{R}$, continuous at the origin, with quadratic supply rate (18.3);

(b) system (18.2) has a quadratic storage function $V(x) = x'Px$ with quadratic supply rate (18.3);

(c) there exist $r > 0$ and $c \in \mathbb{R}$ such that

$$\sum_{t=t_0}^{t_1} \sigma(x(t), w(t)) > c \ \ \forall\ t_1 \geq t_0 \geq 0$$

whenever signals x, w satisfy (18.2) with $|x(t_0)| < r$ and $|x(t_1)| < r$.

If conditions (a)-(c) are satisfied then the limit

$$\Phi(x(\cdot), u(\cdot)) = \lim_{T \to +\infty} \sum_{0}^{T} \sigma(x(t), w(t))dt$$

exists for all $(x, u) \in \mathcal{M}_{DT}(x_0)$*, and the infimum of* Φ *over* $\mathcal{M}_{DT}(x_0)$ *equals* $-x_0' P x_0$ *for some matrix* $P_0 = P_0'$*, which satisfies (18.7), and is also such that* $P_0 \leq P$ *for every* $P = P'$ *satisfying (18.7). Moreover, for the minimal solution* $P = P_0$*, the quadratic form on the left side of the inequality in (18.7) has rank not larger than* m*.*

18.1.1 Proof of Theorem 18.1 (a sketch)

Obviously, condition (b) implies (a). Also, if condition (a) is satisfied then, due to the continuity of V at zero, there exists $r > 0$ such that $|V(x) - V(0)| < 1$ whenever $|x| < r$, hence

$$\int_{t_0}^{t_1} \sigma(x(t), w(t))dt \geq V(x(t_1)) - V(x(t_2)) \geq -2$$

whenever $|x(t_{1,2})| < r$, which means that (a) implies (c).

To prove that (c) implies (b), assume that (c) is valid. Since multiplying a solution (x, w) of (18.1) by a constant yields another solution of (18.1), it follows that

$$\int_{t_0}^{t_1} \sigma(x(t), w(t)) \geq c\delta^2 \quad \forall\ t_1 \geq t_0 \geq 0$$

whenever $|x(t_i)| \leq r\delta^2$. Hence for every pair $(x, u) \in \mathcal{M}_{CT}(x_0)$ and every $\epsilon > 0$ there exists $T > 0$ such that

$$\int_{t_0}^{t_1} \sigma(x(t), w(t)) > -\epsilon \quad \forall\ t_1 > t_0 > T,$$

which proves that the limit $\Phi(x(\cdot), u(\cdot)) \in \mathbb{R} \cup \{+\infty\}$ does exist for all $(x, u) \in \mathcal{M}_{DT}(x_0)$.

Let $V = V(x_0)$ be the maximal lower bound of Φ over $\mathcal{M}_{DT}(x_0)$. Since the pair (A, B) is stabilizable, $V(x_0) < +\infty$. Due to the lower bound provided by (c), $V(x_0) > -\infty$. Since Φ is a quadratic form, $V(x_0)$ is a quadratic form.

Define P_0 by $V(x_0) = -x_0' P_0 x_0$. By the definition of V,

$$V(x(t_0)) \leq V(x(t_1)) + \int_{t_0}^{t_1} \sigma(x(t), w(t))dt$$

for $t_1 \geq t_0$ and for every solution x, w of (18.1) (the inequality says that starting minimization at an earlier time t_0 will bring an outcome not worse than when optimization is postponed until time $t_1 \geq t_0$). Hence, $V(x) = -x_0' P x_0$ is a quadratic storage function of system (18.1) with supply rate σ.

18.2 KYP Lemma In Frequency Domain

The "frequency domain" statements of the KYP Lemma claim that existence of a quadratic storage function with a given quadratic supply rate is equivalent to a frequency domain inequality.

It is easy to check that

$$2\mathrm{Re}\ x'P(Ax + Bw) = 0$$

for every pair of *complex* vectors $x \in \mathbb{C}^n$, $w \in \mathbb{C}^m$ satisfying the relation

$$j\omega x = Ax + Bw \tag{18.9}$$

for some $\omega \in \mathbb{R} \cup \{\infty\}$, where, for $\omega = \infty$, (18.9) is understood as $x = 0$. Since the coefficients of the quadratic form in (18.7) are real, the inequality implies

$$\sigma(x, w) - 2\mathrm{Re}\ x'P(Ax + Bw) \geq 0 \quad \forall\ x \in \mathbb{C}^n,\ w \in \mathbb{C}^m.$$

Therefore (18.7) implies the frequency domain inequality

$$\sigma(x, w) \geq 0 \tag{18.10}$$

for all $x \in \mathbb{C}^n$, $w \in \mathbb{C}^m$ satisfying (18.9) for some $\omega \in \mathbb{R} \cup \{\infty\}$. Similarly, (18.8) implies the frequency domain inequality (18.10) for all $x \in \mathbb{C}^n$, $w \in \mathbb{C}^m$ satisfying

$$zx = Ax + Bw, \quad |z| = 1. \tag{18.11}$$

The following version of the KYP Lemma states that, for controllable state space models, condition (18.10) (subject to (18.9)) is not only necessary but also sufficient for existence of a quadratic storage function.

Theorem 18.3 *Assume that the pair (A, B) is controllable, in the sense that the matrix*

$$M_c = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

has rank n. Then the following conditions are equivalent:

(a) there exists a real matrix $P = P'$ satisfying (18.7);

(b) inequality (18.10) holds for all $x \in \mathbb{C}^n$, $w \in \mathbb{C}^m$ satisfying (18.9) for some $\omega \in \mathbb{R} \cup \{\infty\}$.

The controllability constraint in Theorem 18.3 can be dropped when the inequalities in (18.7) and (18.9) are *strict*.

Theorem 18.4 *The following conditions are equivalent:*

(a) *there exists a real matrix $P = P'$ for which the quadratic form*

$$\sigma(x, w) - 2x'P(Ax + Bw)$$

is strictly positive definite on $\mathbb{R}^n \times \mathbb{R}^m$;

(b) *for every $\omega \in \mathbb{R} \cup \{\infty\}$ the Hermitean form $\sigma(x, w)$ is strictly positive definite on the subspace of all $x \in \mathbb{C}^n$, $w \in \mathbb{C}^m$ satisfying (18.9).*

Theorems 18.3 and 18.4, with equation (18.11) replacing (18.9), remain valid in the discrete time case.