This assignment tests your ability to convert input/output descriptions of systems into state space models.

**Problem 4.1**

A binary signal generator is represented by its behavioral model as the set $S_y = \{ y(\cdot) \}$ of scalar discrete time signals $y = y(t)$ such that $y(t) \in \{-1, 1\}$ for all $t$, and

$$\left| \sum_{t=0}^{T} y(t) \right| < 2 \quad \text{for all } T \in \mathbb{Z}_+.$$  

(a) Represent $S_y$ as the set of all outputs of a minimal state space model, i.e. find a set $X$ with a minimal number of elements, and functions $f_0 : X \times \mathbb{R} \mapsto X$, $g_0 : X \times \mathbb{R} \mapsto \mathbb{R}$ such a sequence $y$ belongs to $S_y$ if and only if it can be represented in the form $y(t) = g_0(x(t), w(t))$ where $x(t) \in X$, $w(t) \in \mathbb{R}$ are sequences satisfying the equations $x(t + 1) = f_0(x(t), w(t))$ for all $t \in \mathbb{Z}_+$.

The problem, as stated, has no solution, because the behavioral model is not time invariant (to see this, note that the signal

$$y_0 = (-1, 1, 1, -1, 1, -1, 1, -1, 1, -1, 1, \ldots)$$

belongs to $S_y$, while its "shifted by 1 step" version

$$y_1 = (1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, \ldots)$$

does not). Since the set of all possible outputs of a state space model

$$x(t + 1) = f_0(x(t), w(t)), \quad y(t) = g_0(x(t), w(t))$$

is always a time-invariant behavior, the problem, as stated, has no answer.

One way of fixing the question is by allowing time varying state space models

$$x(t + 1) = f_0(x(t), w(t), t), \quad y(t) = g_0(x(t), w(t), t),$$

where $f_0 : X \times \mathbb{R} \times \mathbb{Z}_+ \mapsto X$, $g_0 : X \times \mathbb{R} \times \mathbb{Z}_+ \mapsto \mathbb{R}$ are fixed functions. To build such model, it is natural to try using the sum $s(t) = y(0) + \cdots + y(t - 1)$ (where $s(0) = 0$ by definition) in constructing the state $x(t)$. Note that, for even $t$, the sum $s(t)$ must be even, and hence the constraint $|s(t)| < 2$ implies $s(t) = 0$ for even $t$. When $t$ is off, $s(t)$ must be odd as well, hence $|s(t)| < 2$ implies $s(t) \in \{-1, 1\}$. This
suggests defining $X = \{-1, 1\}$ and requiring that $x(t) = s(t)$ when $t$ is odd, while allowing $x(t)$ to take any value in $\{-1, 1\}$ when $t$ is even. This leads to

$$f(x, w, t) = \begin{cases} 
\text{sgn}(w), & t \text{ is even,} \\
1, & t \text{ is odd,}
\end{cases}$$

$$g(x, w, t) = \begin{cases} 
\text{sgn}(w), & t \text{ is even,} \\
-x, & t \text{ is odd,}
\end{cases}$$

where $\text{sgn}(v) = 1$ for $v > 0$ and $\text{sgn}(v) = -1$ for $v \leq 0$. This model uses a state space $X$ with two elements. To show that a single-element $X$ is not enough, note that the value of $y(0)$ determines the value of $y(1)$ completely for all $y \in S_y$ (must have $y(1) = -y(0)$). This effect cannot be produced by a state space model with a single-element $X$, as the set of all possible values for $y(1) = g(x(1), w(1))$ will be the same for all values of $y(0)$, as there is only one possible value for $x(1)$.

(b) Represent $S_y$ as the set of all outputs of a standard state space model

$$x(t + 1) = f(x(t), w(t)), \quad y(t) = g(x(t), w(t)),$$

where $f, g : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are some functions.

As noted in the answer to (a), the problem, as stated, has no solution. When $f, g$ are allowed to be functions of time as well, the solution from (a) can be converted to the standard format by associating an analogue state $x \in \mathbb{R}$ with discrete state $\text{sgn}(x)$, to generate

$$f(x, w, t) = \begin{cases} 
\text{sgn}(w), & t \text{ is even,} \\
1, & t \text{ is odd,}
\end{cases}$$

$$g(x, w, t) = \begin{cases} 
\text{sgn}(w), & t \text{ is even,} \\
-x, & t \text{ is odd,}
\end{cases}$$

Problem 4.2

Professor Geinstein studies the physics of a parallel universe, and believes that the analog of the Newton law $\ddot{y} = u$ should be written as

$$\ddot{y} + \cos(\dot{y}) + \sin(y)\dot{u} = u^2$$

in that universe. He wants to publish his hypothesis in the journal "Gnature", but the journal only accepts state space models with "smooth" right sides, where "smooth" means "differentiable infinitely many times".

(a) Find smooth functions $f : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}^2$ and $g : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}$ such that

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = g(x(t), u(t))$$

is a state space representation of (4.1), in the sense that a pair of smooth signals $u, y$ satisfies (4.1) if and only if it satisfies (4.2) for some signal $x = x(t)$. 

2
The question has many answers. One natural way of approaching the problem is by figuring out what the state variables should be. A guideline here is to try to cover the terms with the highest number of differentiations of \( y \) and \( u \) in the derivative of a single state vector, which suggests using

\[
x_1 = \dot{y} + \sin(y)u, \quad \text{(i.e. } \dot{y} = x_1 - \sin(y)u\text{)}.
\]

Then one can use \( x_2 = y \) to have

\[
\begin{align*}
\dot{x}_1 &= \dot{y} + \sin(y)\dot{u} + \cos(y)\dot{y}u \\
&= u^2 - \cos(\dot{y}) + \cos(y)\dot{y}u \\
&= u^2 + \cos(x_1 - \sin(x_2)u) + \cos(x_2)(x_1 - \sin(x_2)u)u,
\end{align*}
\]

\[
\begin{align*}
\dot{x}_2 &= \dot{y} \\
&= x_1 - \sin(y)u \\
&= x_1 - \sin(x_2)u,
\end{align*}
\]

i.e.

\[
\begin{align*}
f \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, u \right) &= \begin{bmatrix} u^2 + \cos(x_1 - \sin(x_2)u) + \cos(x_2)(x_1 - \sin(x_2)u)u \\ x_1 - \sin(x_2)u \end{bmatrix} \\
g \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, u \right) &= x_2.
\end{align*}
\]

(b) **Professor Gdumb has an alternative hypothesis, and claims that the Newton law should be re-written as \( \ddot{y} = \dot{u}^2 \) for the parallel universe. Is it possible to do for Prof. Gdumb what was done for Prof. Geinstein in (a) (i.e. is it possible to re-write \( \ddot{y} = \dot{u}^2 \) in the state space form (4.2) ?) Explain your answer (a rigorous proof not required).**

There is no way to re-write \( \ddot{y} = \dot{u}^2 \) in an equivalent state space format. Informally speaking, this is due to the fact that a nonlinear transformation is applied to the derivative of \( u \) within the differential equation.

For this particular example, a proof of this can be produced in two steps. First, consider all possible responses \( y = y_k(t) \) to inputs \( u_k(t) = \sin(k^2t) \), where \( k \to \infty \).

Since

\[
\dot{y}_k(t) = k^4 \cos^2(k^2t)
\]

is non-negative and can be made arbitrarily large, the range of \( y(t) \) on any given time interval grows unboundedly with \( k \). The second step is to show that such behavior is not compatible with a standard state space model, as its response to a bounded input cannot grow arbitrarily fast arbitrarily soon for all initial conditions.
To make step 1 formally, note that, due to (4.3), $\dot{y}(t)$ is a monotonic function which is increasing by $\pi k^2$ over every interval of length $2\pi/k^2$. Therefore, on a time interval of length $4\pi/k$, either $\dot{y}(t) \leq -\pi k^2$ in its first quarter, or $\dot{y}(t) \geq \pi k^2$ in its last quarter, which means that the span of $\dot{y}(t)$ over one of those quarters is at least $\pi^2 k$. Hence the span of $y(t)$ over every interval of length $4\pi/k$ is at least $\pi^2 k$ long.

To make step 2 formally, assume to the contrary that a smooth state space model (4.2) matching the equation $\ddot{y} = \dot{u}^2$ exists. Let $y_k, x_k$ be the solutions of (4.2) with $x(0) = 0$ and $u_k = \sin(\frac{k}{2} t)$. Define numbers $c_f, c_g$ according to

$$c_f = \max\{|f(x, u)| : |x| \leq 1, |u| \leq 1\}, \quad c_g = \max\{|g(x, u)| : |x| \leq 1, |u| \leq 1\}$$

(note that $c_f > 0$ as otherwise $y_k = \text{const}$ which we know is not the case). Then $|\dot{x}_k(t)| \leq c_f$ as long as $|x_k(t)| \leq 1$, and hence $|x_k(t)| \leq 1$ and $|y_k(t)| \leq c_g$ for $t \leq 1/c_f$. Since this bound contradicts the arbitrarily fast growth of the span of the values of $y_k(t)$ on an arbitrarily small time interval as $k \to \infty$, the equation $\ddot{y} = \dot{u}^2$ does not have an equivalent state space model.

**Problem 4.3**

Discrete time SISO (single input, single output) system with input $u$ and output $y$ is defined by the equation

$$y(T) = \sum_{t=0}^{T} u(t) \cos(t + T) \quad (T \in \mathbb{Z}_+). \quad (4.4)$$

(a) Find time-dependent matrices $A = A(t), B = B(t), C = C(t), D = D(t)$ such that signals $u, y$ satisfy (4.4) if and only

$$y(t) = C(t)x(t) + D(t)u(t), \quad x(t+1) = A(t)x(t) + B(t)u(t), \quad x(0) = 0 \quad (4.5)$$

for some signal $x$.

Since $\cos(t + T) = \cos(t)\cos(T) - \sin(t)\sin(T)$, we have

$$y(T) = \cos(2T)u(T) + \cos(T)\sum_{t=0}^{T-1} \cos(t)u(t) - \sin(T)\sum_{t=0}^{T-1} \sin(t)u(t),$$

which suggests using the "counter" variables

$$x_1(t) = \sum_{\tau=0}^{t-1} \cos(\tau)u(\tau), \quad x_2(t) = \sum_{\tau=0}^{t-1} \sin(\tau)u(\tau)$$

as two components of $x(t) = [x_1(t); x_2(t)]$. Indeed, since

$$x_1(t+1) = x_1(t) + \cos(t)u(t), \quad x_2(t+1) = x_2(t) + \sin(t)u(t),$$
we get a state space model (4.5) with

\[
A(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, \quad C(t) = [\cos(t), -\sin(t)], \quad D(t) = \cos(2t).
\]

(b) It is usually important to use ”minimal” state space models (referring to the dimension of \( x(t) \)). How would you know that your answer in (a) is a minimal model? To answer the question, for a given large positive integer \( N \) consider the linear transformation \( H_N \) mapping the vector \( u_N = [u(0); u(1); \ldots; u(N - 1)] \) of first \( N \) input samples to the vector \( y_N = [y(N); y(N + 1); \ldots; y(2N - 1)] \) of next \( N \) output samples, assuming that \( u(N) = u(N + 1) = \cdots = u(2N - 1) = 0 \). Find the relation between the rank of \( H_N \) and the dimension of a state space model (4.5), and use it to assess minimality of your answer in (a). (Optional: it would not hurt to make an effort to produce a minimal model if yours is not.)

Subject to equations (4.5), \( x(N) \) is a linear function of \( u_N \), i.e. \( x(N) = L_N^c u_N \) (\( L_N^c \) depends on \( A(t), B(t), C(t), D(t) \) in a relatively complicated way, but this is irrelevant to this discussion). On the other hand, subject to \( u(N) = u(N + 1) = \cdots = u(2N - 1) = 0 \) and equations (4.5), \( y_N \) is a linear function of \( x(N) \), i.e. \( y_N = L_N^o x(N) \). Therefore \( H_N = L_N^o L_N^c \) has rank not larger than the dimension of \( x(N) \).

Function ps4_4.m serves to calculate the rank of \( H_N \) for a given \( N \), verifying the answer using singular values (for numerical rank calculations, \( \text{rank}(M) = k \) means that \( \sigma_{k+1}(M) \) is much smaller than \( \sigma_k(M) \)).

```matlab
function ps4_3(N)
H=cos(repmat(0:N-1,N,1)+repmat((N:(2*N-1))',1,N));
k=rank(H);
s=svd(H);
fprintf(' rank(H)=%d, check: %f >> %f\n',k,s(k),s(k+1))
```

As running ps4_4.m certifies, the rank of \( H_N \) equals 2 for \( N \geq 3 \), which certifies minimality of the state space model derived in (a).