Problem 5.1
This assignment tests your ability to work with matrix exponents to express analytically solutions of system equations.

Consider the state space model

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t), \tag{5.1} \]

where

\[ u(t) = u_0 + u_1 t + u_2 t^2 + u_3 t^3 \tag{5.2} \]

is a polynomial with real coefficients \( u_i \), and \( A, B, C, D \) are constant real matrices of dimensions \( n \)-by-\( n \), \( n \)-by-\( m \), \( k \)-by-\( n \), and \( k \)-by-\( m \) respectively.

(a) Use the standard matrix algebra operations (addition, subtraction, multiplication, concatenation), and the operation of taking the matrix exponent, to express \( y(t) \) as a function of \( A, B, C, D, t, x(0), u_0, u_1, u_2, u_3 \).

(b) Check your answer by writing MATLAB code implementing your algorithm, and comparing its output with the output of \texttt{lsim.m}.

Problem 5.2
Simple instances of evolution matrix computation.

Give an explicit expression for the evolution matrix \( \Phi = \Phi(t, s) \) for the equation

\[ \dot{x}(t) = A(t)x(t) \]

when

(a) \( A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \); 

(b) \( A(t) = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix} \).
Problem 5.3
This assignment tests your ability to use evolution matrices while working on continuous time linear time-varying state space models.

Consider state space model
\[
\dot{x}(t) = A(t)x(t) + u(t), \quad y(t) = x(t)
\] (5.3)
where \( A = A(t) \), \( u = u(t) \) are continuous and such that
\[
A(t + 1) = A(t), \quad u(t) = u(t + 1) \quad \forall \ t \in \mathbb{R}_+.
\] (5.4)

Let \( \Phi = \Phi(t, s) \) be the associated evolution matrix, i.e.
\[
\Phi(t, t) = I, \quad \frac{d\Phi(t, s)}{dt} = A(t)\Phi(t, s) \quad \forall \ t, s.
\] (5.5)

(a) Is it always true that \( \Phi(t + 1, s) = \Phi(t, s) \) for all \( t, s \)? Sketch a proof or give counterexample.

(b) Is it always true that (5.3) has a solution \( x = x(t) \) such that \( x(t) = x(t + 1) \) for all \( t \)? Sketch a proof or give counterexample.

(c) In terms of matrix \( M = \Phi(1, 0) \), give a necessary and sufficient condition for (5.3) to have a unique solution \( x \) such that \( x(t) = x(t + 1) \) for all \( t \) for every continuous \( u = u(t) \) satisfying (5.4).

(d) Assuming the conditions produced in (c) are satisfied, give an explicit formula for the periodic solution \( x(t) = x(t + 1) \) in terms of \( \Phi(\cdot, \cdot) \) and \( u(\cdot) \) (standard matrix algebra operations, as well the operation of integration, are allowed).

(e) For the case when \( A(t) \equiv A_0 \) is constant, express the conditions produced in (c) in terms of the eigenvalues of \( A_0 \).