Problem 7.1

This assignment tests your understanding of discrete time Lyapunov equations.

(a) \( n \)-by-\( n \) real matrices \( A, P = P' \), and \( Q = Q' \) are such that, for the function \( V : \mathbb{R}^n \mapsto \mathbb{R} \) defined by \( V(x) = x'Px \),

\[
V(x) - V(Ax) = x'Qx \quad \forall \ x \in \mathbb{R}^n.
\] (7.1)

Express \( Q \) in terms of \( A \) and \( P \).

(b) Find all continuous functions \( V : \mathbb{R}^2 \mapsto \mathbb{R} \) such that \( V(0) = 0 \) and

\[
V\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) - V\left(\begin{bmatrix} x_1/2 \\ x_2/3 \end{bmatrix}\right) = (x_1 + x_2)^2 \quad \forall \ x_1, x_2 \in \mathbb{R}.
\]

Problem 7.2

This assignment tests your ability to use linearization in assessing local stability of equilibria of autonomous state space models.

For the state space models (a),(b) given below, find all equilibria and classify each equilibrium as either asymptotically stable or unstable (i.e. not stable in the sense of Lyapunov):

(a) CT model:

\[
\begin{align*}
\dot{x}_1(t) &= \cos(x_2(t)), \\
\dot{x}_2(t) &= \cos(x_3(t)), \\
\dot{x}_3(t) &= -x_1(t) + x_2(t) - 2x_3(t).
\end{align*}
\]

(b) DT model:

\[
\begin{align*}
x_1(t + 1) &= x_2(t)^3, \\
x_2(t + 1) &= x_3(t)^3, \\
x_3(t + 1) &= x_1(t)^3.
\end{align*}
\]

Problem 7.3

This assignment tests your ability to use linearization and quadratic Lyapunov functions in a setup which calls naturally for a coordinate-free approach.
For a given real $n$-by-$n$ matrix $A$ consider the linear differential equation
\begin{equation}
\dot{X}(t) = AX(t) + X(t)A',
\end{equation}
(7.2)
to be solved with respect to $n$-by-$n$ Hermitian matrix-valued function $X(t) = X(t)$. Note that equation (7.2) can be written in the form $dv/dt = av$, where $v(t) \in \mathbb{R}^N$ ($N = n^2$) is the vector of coefficients representing $X(t)$ in some basis in the ($n^2$-dimensional) real vector space $H_n$ of all Hermitian $n$-by-$n$ matrices, and $a$ is the $N$-by-$N$ matrix of the linear transformation $X \mapsto AX + XA'$ in this basis. However, such representations are not necessarily helpful in answering the following questions.

(a) Find necessary and sufficient conditions (to be expressed in terms of the set of eigenvalues of $A$) of asymptotic stability of the equilibrium $X = 0$ of (7.2).

(b) (optional, extra credit) Let $P = P' > 0$ be a real $n$-by-$n$ matrix such that $PA + A'P$ is negative definite. Find an explicit expression (using $P$ as a parameter) for a quadratic form $V : H_n \mapsto \mathbb{R}$ such that $V(X(t))$ has negative derivative along all solutions $X(t) \neq 0$ of (7.2).

Hint: if $x = x(t) \in \mathbb{R}^n$ satisfies $dx/dt = Ax$ then $X(t) = x(t)x(t)'$ satisfies (7.2). It is also useful to recall that $x'Px = \text{trace}(Pxx')$ and that $\text{trace}(YW) \geq 0$ when $Y = Y' \geq 0$ and $W = W' \geq 0$ are two positive semidefinite matrices of same dimension.

(c) Consider the nonlinear differential equation
\begin{equation}
\dot{Y}(t) = Q - Y(t)^2,
\end{equation}
(7.3)
where
\[ Q = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 9 & 0 \\
0 & 0 & 0 & 25
\end{bmatrix}, \]
to be solved with respect to 4-by-4 Hermitian matrix-valued function $Y(t) = Y(t)'$. Find all equilibria of (7.3) and classify each equilibrium as either asymptotically stable or unstable.

**Problem 7.4**

For each of the SISO systems (input $w$, output $e$) described below in (a)-(c), calculate their $L_2$ gain as a function of positive integer parameter $k$. Explain your answers.

(a) DT system: $e(t) = \max\{w(\tau) : t \geq \tau \geq t - k\}$.

(b) CT system: $e(t) = w(t/k)$.

(c) DT system: $e(t) = w(t/k)$ when $t/k$ is an integer, $e(t) = 0$ otherwise.