Problem 8.1

This assignment checks your understanding of $L_2$ gain of LTI models.

(a) Find $L_2$ gain of the LTI system defined by its transfer matrix

$$H(s) = \begin{bmatrix} \frac{1}{a+s+s^2} & 0 \\ 0 & 1 \end{bmatrix},$$

where $a \in \mathbb{R}$ is a parameter. Check your answer using MATLAB.

**ANSWER:**

$$L_2 \text{ gain}(H) = \begin{cases} \infty & a \leq 0 \\ \frac{1}{|a|} & 0 < a \leq \frac{1}{2} \\ \frac{1}{\sqrt{a-\frac{1}{4}}} & \frac{1}{2} \leq a \leq \frac{5}{4} \\ 1 & a \geq \frac{5}{4} \end{cases}.$$  

**EXPLANATION:**

For $a \leq 0$ the transfer function $H$ has unstable poles $s = -\frac{1+\sqrt{1-4a}}{2}$ so

$$L_2 \text{ gain}(H) = \infty.$$  

For $a > 0$

$$L_2 \text{ gain}(H) = \sup_{\omega \in \mathbb{R} \cup \{\infty\}} \max \left\{ \frac{1}{\sqrt{(a-\omega^2)^2 + \omega^2}}, 1 \right\}.$$  

The expression $\sqrt{(a-\omega^2)^2 + \omega^2}$ has extrema for $\omega = 0$ and $\omega^2 = \frac{2a-1}{2}$ for which it takes the values $\frac{1}{|a|}$ and $\frac{1}{\sqrt{a-\frac{1}{4}}}$ respectively.

For $0 < a \leq \frac{1}{2}$ we have $1 < \frac{1}{\sqrt{a-\frac{1}{4}}} \leq \frac{1}{|a|}$ so $L_2 \text{ gain}(H) = \frac{1}{|a|}$.

For $\frac{1}{2} \leq a \leq \frac{3}{4}$ we have $\frac{1}{|a|} \leq \frac{1}{\sqrt{a-\frac{1}{4}}}$ and $1 \leq \frac{1}{\sqrt{a-\frac{1}{4}}}$ so $L_2 \text{ gain}(H) = \frac{1}{\sqrt{a-\frac{1}{4}}}$.

For $a \geq \frac{3}{4}$ we have $\frac{1}{|a|} < \frac{1}{\sqrt{a-\frac{1}{4}}} \leq 1$ so $L_2 \text{ gain}(H) = 1$.

(b) Find $L_2$ gain of the LTI system defined by its transfer function

$$H(s) = \frac{1-e^{-as}}{s},$$

where $a \in \mathbb{R}$ is a parameter. Check your answer using MATLAB.
**ANSWER:**

\[
L_2 \text{ gain}(H) = \begin{cases} 
\infty & a < 0 \\
a & a \geq 0
\end{cases}.
\]

**EXPLANATION:**

At first we notice that for \( a < 0 \) the \( L_2 \) gain of \( H \) is \( \infty \). For \( a > 0 \) we expect the \( L_2 \) gain to be finite. The impulse response of the given system is of the form:

\[
h(t)
\]

Using the Laplace transform we get

\[
|H| = \left| \int_{-\infty}^{\infty} e^{-st}h(t)dt \right| \leq \int_{-\infty}^{\infty} |h(t)|dt \Rightarrow L_2 \text{ gain} \leq a.
\]

But for \( a = 0 \) we have \( H(s) = 0 = a \) so \( L_2 \) gain(H) = a.

(c) Two LTI systems defined by transfer matrices \( H_1 = H_1(s) \) and \( H_2 = H_2(s) \) both have \( L_2 \) gain of 2. What is the possible range for the \( L_2 \) gain of the LTI system defined by \( H = H_1H_2 \)? Explain your answer.

**ANSWER:**

\[
0 \leq L_2 \text{ gain}(H) \leq 4
\]

**EXPLANATION:**

From the submultiplicative property we know that:

\[
L_2 \text{ gain}(H) = L_2 \text{ gain}(H_1H_2) \leq L_2 \text{ gain}(H_1)L_2 \text{ gain}(H_2) = 2 \times 2 = 4.
\]

We also know that the \( L_2 \) gain is always a nonnegative number, so \( L_2 \) gain(H) \( \geq 0 \).
Using $H_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $H_2 = \begin{bmatrix} r & 0 \\ 0 & 2 \end{bmatrix}$, $r \in [0,2]$ we see that $L_2$ gain(H) can take any value in $[0,4]$.

(d) Two LTI systems defined by transfer matrices $H_1 = H_1(s)$ and $H_2 = H_2(s)$ have $L_2$ gains of 1 and 3 respectively. What is the possible range for the $L_2$ gain of the LTI system defined by $H = H_1 + H_2$? Explain your answer.

**ANSWER:**

$$2 \leq L_2 \text{ gain}(H) \leq 4$$

**EXPLANATION:**

Using the triangular inequality we see that:

$L_2 \text{ gain}(H) = L_2 \text{ gain}(H_1 + H_2) \leq L_2 \text{ gain}(H_1) + L_2 \text{ gain}(H_2) = 3 + 1 = 4$

and

$L_2 \text{ gain}(H) = L_2 \text{ gain}(H_1 + H_2) \geq L_2 \text{ gain}(H_1) - L_2 \text{ gain}(H_2) = 3 - 1 = 2$

Using $H_1 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$ and $H_2 = \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}$, $r \in [-1,1]$ we see that $L_2$ gain(H) can take any value in $[2,4]$.

**Problem 8.2**

This assignment tests your ability to work with quadratic dissipation inequalities in LTI state space models.

For the state space model

$$\dot{x}(t) = ax(t) + w(t), \quad e(t) = w(t) + x(t), \quad (8.1)$$

where $a \in \mathbb{R}$ is a parameter, we are interested in certifying that the $L_2$ gain (from $w(\cdot)$ to $e(\cdot)$) does not exceed $\sqrt{2}$ by finding a function $V : \mathbb{R} \mapsto [0,\infty)$ such that

$$V(x(c)) - V(x(b)) \leq \int_b^c \{2|w(t)|^2 - |w(t) + x(t)|^2\} dt \quad (8.2)$$

for every solution of (8.1) and all $c \geq b \geq 0$. 

(a) Find all values \( a \in \mathbb{R} \) for which L2 gain of the system (input \( w \), output \( e \)) defined by state space model (8.1) is not larger than \( \sqrt{2} \).

**ANSWER:**
\[ a \leq -1 - \sqrt{2} \]

**EXPLANATION:**
\[
H(s) = 1 + \frac{1}{s - a}.
\]

For \( a \geq 0 \) the L2 gain of \( H \) is \( \infty \).

For \( a < 0 \) we have \( L2 \text{ gain}(H) = \sup_{\omega \in \mathbb{R}} H(j\omega) = \sqrt{1 + \frac{1 - 2a}{a^2}} \). The requirement for \( L2 \text{ gain}(H) \leq \sqrt{2} \) is satisfied by \( a \leq -1 - \sqrt{2} \).

(b) Find the set \( A \) of all values of \( a \) for which a function \( V : \mathbb{R} \to \mathbb{R} \) satisfying (8.2) exists.

**ANSWER:**
\[ A = (-\infty, -1 - \sqrt{2}] \cup [-1 + \sqrt{2}, \infty) \]

**EXPLANATION:**
Let \( V(x) = px^2 \) some quadratic function. Then \( V(x) \) satisfies (8.2) iff
\[
\dot{V}(x) = 2px(ax + w) \leq 2w^2 - (w + x)^2 \Rightarrow
\]
\[
[w \quad x]^{\prime} \begin{bmatrix} 1 & -p - 1 \\ -p - 1 & -2pa - 1 \end{bmatrix} [w \quad x] \geq 0 \Rightarrow
\]
\[
det \begin{bmatrix} 1 & -p - 1 \\ -p - 1 & -2pa - 1 \end{bmatrix} \geq 0 \text{ (Sylvester’s criterion) } \Rightarrow
\]
\[
p^2 + 2p(a + 1) + 2 \leq 0
\]
which can only be realized if the polynomial has two different real solutions. This only occurs when \( 4(a + 1)^2 - 8 \geq 0 \Rightarrow a \in \bar{A} = (-\infty, -1 - \sqrt{2}] \cup [-1 + \sqrt{2}, \infty) \).

So, we have shown that \( \forall a \in \bar{A} = (-\infty, -1 - \sqrt{2}] \cup [-1 + \sqrt{2}, \infty) \) there exists some quadratic function of the form \( V(x) = px^2 \) that satisfies (8.2). However these may not be the only values of \( a \) for which some arbitrary (and not necessarily quadratic) function satisfying (8.2) exists. In other words \( \bar{A} \subset A \). Let \( a \not\in A \Rightarrow a \in (-1 - \sqrt{2}, -1 + \sqrt{2}) \). Then if we choose \( w(t) = -ax(t) \) the state evolution equation becomes \( \dot{x} = 0 \Rightarrow x(b) = x(c) \) for \( b \neq c \). Substituting in (8.2) we arrive to \( 0 \leq \int_b^c (2a^2 - (1 - a)^2)x^2dt \), which is a contradiction because \( 2a^2 - (1 - a)^2 < 0 \), \( \forall a \in (-1 - \sqrt{2}, -1 + \sqrt{2}) \). Hence \( A = \bar{A} = (-\infty, -1 - \sqrt{2}] \cup [-1 + \sqrt{2}, \infty) \).
(c) For all \( a \in A \) find a function \( V : \mathbb{R} \mapsto \mathbb{R} \) satisfying (8.2).

**ANSWER:**

\[ V(x) = -(a + 1)x^2 \]

**EXPLANATION:**

Let \( p_o = -a - 1 \) the value of \( p \) that minimizes \( p^2 + 2p(a + 1) + 2 \) (from part b)). Then \( V(x) = p_ox^2 \) satisfies (8.2).

(d) Find the set \( A_+ \) of all values of \( a \) for which a function \( V : \mathbb{R} \mapsto [0, \infty) \) satisfying (8.2) exists.

Explain the relation between the answers in (a)-(d).

**ANSWER:**

\[ A_+ = \{(-\infty, -1 - \sqrt{2}] \}

**EXPLANATION:**

Given \( p_o \) from (c) we want \( p_o \geq 0 \Rightarrow a \leq -1. \) So \( \forall a \in (-\infty, -1 - \sqrt{2}] \) the nonnegative function \( V(x) = -(a + 1)x^2 \) satisfies (8.2). Thus \( A_+ = (-\infty, -1 - \sqrt{2}] \).

Problem 8.3

This assignment tests your understanding of interconnections of systems.

Let \( \mathcal{G} \) be the system (input \( w \), output \( e \)) defined by the feedback interconnection

\[ \begin{array}{c}
\text{\( w \)} \\
\downarrow \\
\mathcal{H} \\
\uparrow \\
\text{\( y \)} \\
\downarrow \\
\text{\( u \)} \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\downarrow \\
\text{\( e \)} \\
\end{array} \]

where \( \mathcal{H} \) is the system defined by the state space model

\[ \dot{x} = Ax + Bu, \quad y = Cx + Du, \]

and \( A, B, C, D \) are real matrices of dimensions \( n \)-by-\( n \), \( n \)-by-\( m \), \( m \)-by-\( n \), and \( m \)-by-\( m \) respectively.
(a) In terms of the eigenvalues of $D$, give necessary and sufficient conditions for the feedback interconnection to be well-posed (i.e. for $G$ to be a causal input/output map).

**ANSWER:**

$\text{eig}(D) \neq 1$

**EXPLANATION:**

Given the above system, $u = e = w + Cx + Du \Rightarrow (I - D)u = w + Cx$. For the system to be well-posed we need $u$ to be uniquely defined by $w$. Thus we need $I - D$ to be invertible, which is true iff $\text{eig}(D) \neq 1$.

(b) Assuming the conditions derived in (a) are satisfied, find explicit expressions (in terms of $A, B, C, D$) for the coefficient matrices of a state space model of $G$.

**ANSWER:**

\[
\begin{align*}
\bar{A} &= A + B(I - D)^{-1}C \\
\bar{B} &= B(I - D)^{-1} \\
\bar{C} &= (I - D)^{-1}C \\
\bar{D} &= (I - D)^{-1}
\end{align*}
\]

**EXPLANATION:**

\[
e = w + Cx + De \Rightarrow e = (I - D)^{-1}Cx + (I - D)^{-1}w
\]

\[
\dot{x} = Ax + Bu = Ax + Be = (A + B(I - D)^{-1}C)x + B(I - D)^{-1}w.
\]

(c) Assuming the conditions derived in (a) are satisfied, express the transfer matrix $G = G(s)$ of system $G$ in terms of the transfer matrix $H = H(s)$ of system $H$.

**ANSWER:**

\[
G(s) = (I - H(s))^{-1}
\]

**EXPLANATION:**

\[
E(s) = W(s) + Y(s) = W(s) + H(s)E(s) \Rightarrow \\
(I - H(s))E(s) = W(s) \Rightarrow \\
E(s) = (I - H(s))^{-1}W(s) \Rightarrow \\
G(s) = (I - H(s))^{-1}
\]


Function ps8_1a(n)

```matlab
function ps81a(n)

if nargin<1, n=50; end
s=tf('s');

a=2*(1:n)/n;
ga=(a<0.5)./a+(a>=0.5).*(a<1.25)./sqrt(a-0.25)+(a>=1.25);
gn=zeros(1,n);
for k=1:n,
    gn(k)=norm([1/(a(k)+s+s^2) 0;0 1],Inf);
end
close(gcf);semilogy(a,ga,a,gm,'.');grid
```

Output

![Figure No. 1](image-url)