Problem 9.1
This assignment asks you to use the small gain theorem in the analysis of a nonlinear model.
Consider the system (with input $w = w(t)$ and output $e = e(t)$) described by the state space model
\[
\dot{x}_1(t) = -x_1(t) + x_2(t), \quad \dot{x}_2(t) = -x_2(t) + ax_1(t)\phi(t, x_1(t)) + w(t), \quad e(t) = x_1(t),
\]
where $\phi : \mathbb{R} \times \mathbb{R} \mapsto [0, 1]$ is a continuous function, and $a \in \mathbb{R}$ is a parameter. The exact form of $\phi(\cdot, \cdot)$ is not assumed to be known.

Find all values of $a$ for which the small gain theorem can be used to establish a finite upper bound for the $L^2$ gain of the system, and give an explicit expression for such upper bound.

Problem 9.2
This assignment tests your ability to use feedback in representing models in a format most suitable for the analysis using the small gain theorem and its generalizations.
Consider linear time varying system described by equations
\[
\frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{dt^2} + a(t)\frac{dy(t)}{dt} + a(t)^3y(t) = a(t)^2u(t),
\]
where $a : \mathbb{R} \mapsto [1 - r, 1 + r]$ is a given continuous function, and $r > 0$ is a parameter. The exact form of $a(\cdot)$ is not assumed to be known.

(a) Re-write equations (9.1) in the form
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_1w(t) + B_2u(t), \\
y(t) &= Lx(t), \\
e(t) &= Cx(t) + D_1w(t) + D_2u(t), \\
w(t) &= \delta(t)e(t),
\end{align*}
\]
where
\[
x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix}, \quad \delta(t) = \frac{a(t) - 1}{r},
\]
and the matrices $A, B_i, C, D_i$ do not depend on $a(\cdot)$ (dependence on $r$ is admissible).
(b) Find maximal possible value of the L2 gain of the system mapping $n$-dimensional signal $v = v(t)$ to $n$-dimensional signal $z = z(t)$ according to $z(t) = \delta(t)v(t)$ where $\delta : \mathbb{R} \mapsto [-1, 1]$ is a continuous function. What are the values of the H-Infinity norm of the transfer matrix

$$G(s) = D_1 + C(sI - A)^{-1}B_1,$$

where $A, B_i, C, D_i$ are defined in (a), for which the small gain theorem guarantees finiteness of the L2 gain (from $u$ to $y$) of system (9.1)? Find numerically the largest value of $r$ for which this argument proves stability of system (9.1).

(c) Describe the set of all $n$-by-$n$ matrices $Q = Q'$ and $R = R'$ such that the inequality

$$\int_0^T \{v(t)'Qv(t) - z(t)'Rz(t)\} dt \geq 0$$

holds for all $n$-dimensional signals $v, z$ satisfying the relation defined in (b).

(d) Use the result of (c) to define a family of quadratic forms $\sigma = \sigma(w, e)$ such that positive definiteness of the quadratic form

$$\gamma^2|u|^2 - |Lx|^2 + \sigma(w, Cx + D_1w + D_2u) - 2x'P(Ax + B_1w + B_2u) \quad (9.2)$$

for some $P = P' \geq 0$ implies that $\gamma \geq 0$ is an upper bound for the L2 gain from $u$ to $y$ in system (9.1).

(e) According to the KYP lemma, positive definiteness of the quadratic form (9.2) constructed in (e) is equivalent to the inequality $\|G_\sigma\|_\infty < 1$, where $G_\sigma$ is a transfer matrix ($G_\sigma$ depends on $r$, $\gamma$, and the coefficients of $\sigma$). Give an explicit expression for $G_\sigma$, and use it to search randomly for a better bound of the L2 gain from $u$ to $y$ in system (9.1).