Problem 9.1

This assignment asks you to use the small gain theorem in the analysis of a nonlinear model.

Consider the system (with input $w = w(t)$ and output $e = e(t)$) described by the state space model

$$\dot{x}_1(t) = -x_1(t) + x_2(t), \quad \dot{x}_2(t) = -x_2(t) + ax_1(t)\phi(t,x_1(t)) + w(t), \quad e(t) = x_1(t),$$

where $\phi : \mathbb{R} \times \mathbb{R} \mapsto [0, 1]$ is a continuous function, and $a \in \mathbb{R}$ is a parameter. The exact form of $\phi(\cdot, \cdot)$ is not assumed to be known.

Find all values of $a$ for which the small gain theorem can be used to establish a finite upper bound for the $L_2$ gain of the system, and give an explicit expression for such upper bound.

**ANSWER:**

$$\|G\|_\infty \leq \max\left\{\frac{1-\frac{1}{2}a^2}{\frac{1}{4}\max\{\frac{1}{1-2a}, \frac{1}{\sqrt{2}}\}}\right\}$$

for $-8 < a < 1$

**EXPLANATION:**

The continuous function $\phi$ can be written in the form $\phi(t, x_1(t)) = \frac{1}{2} + \delta(t, x_1(t))$, where the first term of the sum is the approximation we assume for $\phi$ and $\delta$ the approximation error with $\|\delta\|_\infty \leq \frac{1}{2}$. Similarly we can rewrite our state space model:

$$\begin{align*}
\dot{x}_1(t) &= -x_1(t) + x_2(t) \\
\dot{x}_2(t) &= -x_2(t) + \frac{a}{2}x_1(t) + a\delta x_1(t) + w(t) \\
e(t) &= x_1(t)
\end{align*}$$

Given that our system includes uncertainty we can represent it as follows:

![System Diagram]

where $M$ has state space model:
\[
\dot{x}(t) = \begin{bmatrix}
-1 & 1 \\
\frac{a}{2} & -1
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
1
\end{bmatrix} (w(t) + z(t)) + e(t) = \begin{bmatrix}
1 & 0
\end{bmatrix} x(t)
\]

and \( \Delta = a \delta \). The transfer function of \( M \) is 
\[
M(s) = \frac{1}{s^2 + 2s + a^2 - 1 - \frac{a^2}{2}}.
\]

We want the nominal (without the uncertainty of \( \phi \)) system to be stable, which requires 
\( a < 2 \) in order for the poles of \( M(s) \) to lie in the left half plane. The \( L_2 \) gain of \( M \)
\[
\| M \|_\infty = \sup_\omega |M(j\omega)|
\]
is computed as follows:
\[
|M(j\omega)|^2 = M(j\omega)M(-j\omega) = \frac{1}{(\omega^2 - 1 + \frac{a^2}{2})^2 + 4\omega^2}.
\]
The above expression has extrema for \( \omega = 0 \) and \( \omega^2 = -1 - \frac{a^2}{2} \) the values \( \frac{1}{(1-r^2)} \) and \( \frac{1}{\sqrt{2a}} \) respectively. So \( \| M \|_\infty = \max\{\frac{1}{(1-r^2)}, \frac{1}{\sqrt{2a}}\} \). It is also obvious that \( \| \Delta \|_\infty \leq \frac{|a|}{2} \). For the small gain theorem to hold we want both 
\( |\frac{|a|}{2}| \frac{1}{(1-r^2)} | < 1 \) and \( |\frac{|a|}{2}| \frac{1}{\sqrt{2a}} | < 1 \) which reduce to 
\( -8 < a < 1 \).

The transfer function of our system is 
\[
G = \frac{M}{1-M\Delta}\] so the \( L_2 \) gain of the system is given by:
\[
\| G \|_\infty = \| M \|_\infty \| M \|_\infty \| \Delta \|_\infty \leq \frac{\max\{\frac{1}{(1-r^2)}, \frac{1}{\sqrt{2a}}\}}{1 - \frac{|a|}{2} \max\{\frac{1}{(1-r^2)}, \frac{1}{\sqrt{2a}}\}}.
\]

**Problem 9.2**

This assignment tests your ability to use feedback in representing models in a format most suitable
for the analysis using the small gain theorem and its generalizations.

Consider linear time varying system described by equations
\[
\frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{dt^2} + a(t)\frac{dy(t)}{dt} + a(t)^3y(t) = a(t)^2u(t),
\]
where \( a : \mathbb{R} \rightarrow [1 - r, 1 + r] \) is a given continuous function, and \( r > 0 \) is a parameter.
The exact form of \( a(\cdot) \) is not assumed to be known.

(a) Re-write equations (9.1) in the form
\[
\dot{x}(t) = Ax(t) + B_1w(t) + B_2u(t),
\]
\[
y(t) = Lx(t),
\]
\[
e(t) = Cx(t) + D_1w(t) + D_2u(t),
\]
\[
w(t) = \delta(t)e(t),
\]
where

\[
x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}, \quad \delta(t) = \frac{a(t) - 1}{r},
\]

and the matrices \( A, B, C, D \) do not depend on \( a(\cdot) \) (dependence on \( r \) is admissible).

**ANSWER:**

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3r^2 & -3r & -r^3 & -r & 2r & r^2 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t)
\]

\[
e(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)
\]

\[
w(t) = \delta(t)e(t) = \begin{bmatrix} \delta(t)x_1 \\ \delta^2(t)x_1 \\ \delta^3(t)x_1 \\ \delta(t)x_2 \\ \delta(t)u(t) \\ \delta^2(t)u(t) \end{bmatrix}.
\]

**EXPLANATION:**

We are given that \( \delta(t) = \frac{a(t) - 1}{r} \) so \( a(t) = r\delta(t) + 1 \). We choose the states of our system to be

\[
x_1 = y(t)
\]
\[
x_2 = \dot{y}(t)
\]
\[
x_3 = \ddot{y}(t)
\]

hence the state space model is given by:

\[
\dot{x}_1(t) = x_2(t)
\]
\[
\dot{x}_2(t) = x_3(t)
\]
\[
\dot{x}_3(t) = -2x_3(t) - a(t)x_2(t) - a^3(t)x_1(t) + a^2(t)u(t).
\]

Plugging in \( \alpha(t) = r\delta(t) + 1 \) we get the answer.

(b) Find maximal possible value of the \( L_2 \) gain of the system mapping \( n \)-dimensional signal \( v = v(t) \) to \( n \)-dimensional signal \( z = z(t) \) according to \( z(t) = \delta(t)v(t) \) where \( \delta : \mathbb{R} \mapsto [-1, 1] \) is a continuous function. What are the values of the \( H\)-Infinity norm of the transfer matrix

\[
G(s) = D_1 + C(sI - A)^{-1}B_1,
\]

where \( A, B_i, C, D_i \) are defined in (a), for which the small gain theorem guarantees finiteness of the \( L_2 \) gain (from \( u \) to \( y \)) of system (9.1)? Find numerically the largest value of \( r \) for which this argument proves stability of system (9.1).

**ANSWER:**

\[ ||G||_\infty < 1 \]

**EXPLANATION:**

Given \( z(t) = \delta(t)v(t) \) and \( \delta : \mathbb{R} \mapsto [-1, 1] \) we get:

\[
\int_0^T |z^2(t)|\, dt = \int_0^T |\delta^2(t)||v^2(t)|\, dt \leq \int_0^T |v^2(t)|\, dt \Rightarrow L_2\text{gain}_{v \rightarrow z} \leq 1.
\]

Our system is uncertain so we can represent it as follows:
where $M$ the nominal system and $\Delta = \delta$ represents the uncertainty. $G(s)$ is the transfer function from $w(t)$ to $e(t)$. For the small gain theorem to hold and consequently the L2 gain of the entire system to be finite we need $\|\Delta\|_{\infty}\|G\|_{\infty} < 1$. But from the first part of b) we know that $\|\Delta\|_{\infty} \leq 1$ so the H-infinity norm of $G$ should be $\|G\|_{\infty} < 1$.

(c) Describe the set of all $n$-by-$n$ matrices $Q = Q'$ and $R = R'$ such that the inequality

$$\int_0^T \{v(t)'Qv(t) - z(t)'Rz(t)\}dt \geq 0$$

holds for all $n$-dimensional signals $v, z$ satisfying the relation defined in (b).

**ANSWER:**
$Q \geq R$ and $Q \geq 0$

**EXPLANATION:**
For $z(t) = \delta(t)v(t)$ we get:

$$\int_0^T \{v(t)'Qv(t) - z(t)'Rz(t)\}dt \geq 0, \forall v, z \Rightarrow$$

$$\int_0^T \{v(t)'Qv(t) - \delta^2(t)v(t)'Rv(t)\}dt \geq 0, \forall v \Rightarrow$$

$$v(t)'Qv(t) - \delta^2(t)v(t)'Rv(t) \geq 0, \forall v \Rightarrow$$

$$Q - \delta^2(t)R \geq 0 \Rightarrow$$

$$Q \geq \delta^2(t)R \geq 0 \Rightarrow$$

$$Q \geq R$$ and $Q \geq 0$.

(d) Use the result of (c) to define a family of quadratic forms $\sigma = \sigma(w, e)$ such that positive definiteness of the quadratic form

$$\gamma^2|u|^2 - |Lx|^2 + \sigma(w, Cx + D_1w + D_2u) - 2x'P(Ax + B_1w + B_2u)$$

(9.2)

for some $P = P' \geq 0$ implies that $\gamma \geq 0$ is an upper bound for the L2 gain from $u$ to $y$ in system (9.1).

**ANSWER:**
$\sigma(w, Cx + D_1w + D_2u) = w(t)'Rw(t) - e(t)'Qe(t)$

**EXPLANATION:**
We are given that:

$$\gamma^2|u|^2 - |Lx|^2 + \sigma(w, Cx + D_1w + D_2u) - 2x'P(Ax + B_1w + B_2u) > 0 \Rightarrow$$
∫_0^T [γ^2|u|^2 - |Lx|^2 + σ(w, Cx + D_1 w + D_2 u) - 2x'P(Ax + B_1 w + B_2 u)]dt > 0.

Let V(x) = x'Px ≥ 0 (P = P' ≥ 0) a quadratic function. Then

∫_0^T [γ^2|u|^2 - |Lx|^2 + σ(w, Cx + D_1 w + D_2 u)]dt ≥ V(x(T)) - V(x(0)) ≥ -V(x(0)).

This holds ∀x, u, w, hence it also holds for x(0)=0. Thus,

∫_0^T [γ^2|u|^2 - |Lx|^2 + σ(w, Cx + D_1 w + D_2 u)]dt ≥ 0.

Choosing σ(w, Cx + D_1 w + D_2 u) = w(t)'Rw(t) - e(t)'Qe(t) with Q ≥ R, Q ≥ 0, and using the result of (c) we compute

∫_0^T [γ^2|u|^2 - |Lx|^2]dt ≥ 0

which implies that γ ≥ 0 is an upper bound for the L2 gain from u to y in the original system.

(e) According to the KYP Lemma, positive definiteness of the quadratic form (9.2) constructed in (e) is equivalent to the inequality ∥G_σ∥_∞ < 1, where G_σ is a transfer matrix (G_σ depends on r, γ, and the coefficients of σ). Give an explicit expression for G_σ, and use it to search randomly for a better bound of the L2 gain from u to y in system (9.1).

**ANSWER:**

\[
G_σ(s) = \begin{bmatrix}
\frac{1}{γ}L(sI - A)^{-1}B_2 & L(sI - A)^{-1}B_1(R^\frac{1}{2})^{-1} \\
\frac{1}{γ}Q^\frac{1}{2}[C(sI - A)^{-1}B_2 + D_2] & Q^\frac{1}{2}[C(sI - A)^{-1}B_1 + D_1](R^\frac{1}{2})^{-1}
\end{bmatrix}
\]

**EXPLANATION:**

According to the KYP Lemma, for some system \( \dot{x} = Ax + Bu \) and for given quadratic form \( σ(x, u) = γ^2∥u∥^2 - ∥Cx + Du∥^2 \) the two following facts are equivalent:

a) \( ∃P = P' : σ(x, u) - 2x'P(Ax + Bu) > 0 \)

b) \( ∃V(x) : σ(x, u) > V(x(t)) \) and \( σ(x, u) > 0 \) on the subset \( (x, u) : jωx = Ax + Bu, \) \( ω ∈ ℝ ∪ ∞. \)

But on the subset \( (x, u) : jωx = Ax + Bu \) we get

\( σ(x, u) = γ^2∥u∥^2 - ∥Cx + Du∥^2 = γ^2∥u∥^2 - ∥G(jω)u∥^2. \)
So according to the KYP Lemma, if $\exists P = P' : \sigma(x,u) - 2x'P(Ax + Bu) > 0$ then $G(j\omega) < \gamma$.

We are given in part d) that

$$\gamma^2 |u|^2 - |Lx|^2 + w(t)'Rw(t) - e(t)'Qe(t) - 2x'P(Ax + B_1w + B_2u) > 0.$$ 

Then, based on the above this is equivalent to $G_\sigma(j\omega) < 1$ where

$$G_\sigma(s) = \begin{bmatrix}
\frac{1}{\gamma}L(sI - A)^{-1}B_2 & L(sI - A)^{-1}B_1(R_2^{1/2})^{-1} \\
\frac{1}{\gamma}Q_2^{1/2}[C(sI - A)^{-1}B_2 + D_2] & Q_2^{1/2}[C(sI - A)^{-1}B_1 + D_1](R_2^{1/2})^{-1}
\end{bmatrix}$$

the transfer matrix from $u_\sigma = \begin{bmatrix} \gamma u \\ R_2^{1/2}w \end{bmatrix}$ to $y_\sigma = \begin{bmatrix} y \\ Q_2^{1/2}e \end{bmatrix}$. 