Problem Q2S1

Real 2-by-1 vector $x(t)$ satisfies the differential equation

$$\dddot{x} + \ddot{x} + Ax = te^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x(0) = \dot{x}(0) = 0$$

where $A$ is a 2-by-2 real matrix. Express $x(2)$ as a function of $A$ using the operations of matrix exponentiation, addition, concatenation, and multiplication.

**Hint:** $\dddot{x} + \ddot{x} + Ax = Bu$ has state space model

$$\dot{v} = \begin{bmatrix} 0 & I_2 \\ -A & -I_2 \end{bmatrix} v + \begin{bmatrix} 0 \\ B \end{bmatrix} u,$$

and $te^t = C_0 \exp(A_0 t)B_0$, where

$$C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Problem Q2S2

Is it true that $e^{A+B} = e^A e^B$ for arbitrary real $n$-by-$n$ matrices $A$, $B$? Sketch a proof or give a counterexample.

**Hint:** the identity should fail for almost any pair of non-commuting matrices. So, try any two non-commuting 2-by-2 matrices for which the exponents are easy to calculate.

Problem Q2S3

$\Phi = \Phi(t, s)$ is the evolution matrix of system

$$\dot{x}(t) = A(t)x(t),$$

where $A = A(t)$ is a fixed continuous 2-by-2 real matrix-valued 3-periodic function of time $t \in \mathbb{R}$. It is known that

$$\Phi(1, 4) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Find the trace of $\Phi(-1, 7)\Phi(2, 1)^{-1}$. 

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Hint: use the identities

\[
\begin{align*}
\text{trace}[AB] &= \text{trace}[BA], \\
\Phi(t,s)^{-1} &= \Phi(s,t), \\
\Phi(t,s) &= \Phi(t,\tau)\Phi(\tau,s), \\
\Phi(t,s) &= \Phi(t+3,s+3).
\end{align*}
\]

Problem Q2S4

Find an explicit formula for

\[
\exp\left[ \begin{array}{ccc} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{array} \right]
\]

as a function of real parameters \(a, b\).

Hint: use

\[
e^{A} = \sum_{k=0}^{\infty} \frac{A^k}{k!}
\]

to calculate \(\exp(A)\) for matrices \(A\) such that \(A^m = 0\) for some \(m\). The identity \(\exp(rI + A) = e^r\exp(A)\) will be helpful as well.

Problem Q2S5

Find explicitly the evolution matrix of the state space model \(\dot{x}(t) = A(t)x(t)\) where

\[
A(t) = A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ for } t < 1, \quad A(t) = A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ for } t \geq 1.
\]

Hint: \(\Phi(t,s) = e^{(t-s)A_0}\) for \(t, s \leq 1\), \(\Phi(t,s) = e^{(t-s)A_1}\) for \(t, s \geq 1\).

Problem Q2S6

State space model

\[
\dot{x}(t) = A(t)x(t) + u(t), \quad (2.1)
\]

where \(A = A(t)\) is a 3-by-3 continuous real matrix-valued function, has evolution matrix \(\Phi = \Phi(t,s)\).

(a) Find necessary and sufficient conditions, in terms of \(\Phi(\cdot,\cdot)\), which guarantee that, for every signal \(u = u(t)\), equation (2.1) has a solution \(x = x(t)\) such that \(x(0) + x(1) = 0\).
(b) **Assuming the condition derived in (a) is satisfied, express the solution of (2.1) such that** \( x(0) + x(1) = 0 \) **in terms of** \( \Phi(t, \cdot) \) and \( u(\cdot) \), **using the operations of integration, matrix addition, subtraction, and multiplication.**

**Hint:** use the ”solution” formula

\[
x(t) = \Phi(t, s)x(s) + \int_s^t \Phi(t, \tau)u(\tau)d\tau.
\]

**Problem Q2S7**

**Autonomous discrete time system is defined by equation** \( x(t + 1) = f(x(t)) \), where \( x(t) \in X = \{0, 1, 2, 3\} \), \( f(0) = 1 \), \( f(1) = 2 \), \( f(2) = 1 \), and \( f(3) = 2 \). **The map** \( V : X \mapsto \mathbb{R} \) **is a Lyapunov function for the system, in the sense that** \( V(x(t)) \) **is not increasing along the solutions of** \( x(t + 1) = f(x(t)) \). **If** \( V(0) = 3 \) **and** \( V(3) = 0 \), **what are the possible values of** \( V(1) \)?

**Hint:** \( V : X \mapsto \mathbb{R} \) **is a Lyapunov function for the system if and only if**

\[
V(0) \geq V(1), \quad V(1) \geq V(2), \quad V(2) \geq V(1), \quad V(3) \geq V(2).
\]

**Problem Q2S8**

**Continuous time state space model**

\[
\dot{x}(t) = f(x(t), w(t)), \quad e(t) = g(x(t), w(t)) \tag{2.2}
\]

**is defined by continuous functions** \( f : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}^2 \) **and** \( g : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \)** **satisfying the inequality**

\[
\dot{V}(x)f(x, w) \leq |w| - |g(x, w)| \quad \forall \ w \in \mathbb{R}, \ x \in \mathbb{R}^2
\]

**for some continuously differentiable function** \( V : \mathbb{R}^2 \mapsto [-1, 1] \). **Assuming that** \((w, e)\) **is an input/output pair of system (2.2) such that**

\[
\int_0^\infty |w(t)|dt \leq 5,
\]

**what are the possible values of**

\[
\int_0^\infty |e(t)|dt?
\]

**Hint:** integrate the dissipation inequality from zero to \( T \), where \( T \to \infty \), and use the fact that \( V(a) - V(b) \leq 2 \) **for all** \( a, b \in \mathbb{R} \).
Problem Q2S9

CONTINUOUSLY DIFFERENTIABLE FUNCTIONS $f: \mathbb{R}^3 \mapsto \mathbb{R}^3$ AND $V: \mathbb{R}^3 \mapsto \mathbb{R}$ ARE SUCH THAT $\dot{V}(a)f(a) \leq 0$ FOR ALL $a \in \mathbb{R}^3$, AND $V(0) > V(a)$ FOR ALL $a \in \mathbb{R}^3$, $a \neq 0$.

(a) WHAT ARE THE POSSIBLE VALUES OF $f(0)$?

**Hint:** look at $V(q(t))$ the "backward in time" solutions of $dx/dt = f(x)$, i.e. the solutions of $dq/dt = -f(q)$.

(b) DOES THE ASSUMPTION IMPLY THAT $x = 0$ IS NOT A STABLE (IN THE SENSE OF LYAPUNOV) EQUILIBRIUM OF THE AUTONOMOUS STATE SPACE MODEL $dx/dt = f(x)$? SKETCH A PROOF OR GIVE A COUNTEREXAMPLE.

**Hint:** the assumptions do not indicate that solutions of $dx/dt = f(x)$ move away from zero.

Problem Q2S10

FIND ALL CONTINUOUS FUNCTIONS $V: \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ SUCH THAT $V(1, 1) = 0$ AND

$$\frac{dV(y(t), \dot{y}(t))}{dt} = y(t)^2$$

FOR EVERY FUNCTION $y: \mathbb{R} \mapsto \mathbb{R}$ SATISFYING THE DIFFERENTIAL EQUATION

$$\ddot{y}(t) - \dot{y}(t) + y(t) = 0.$$ 

**Hint:** this is about using Lyapunov equations to find all functions $V = V(x)$ such that $dV(x(t)) = x(t)'Qx(t)$ for all solutions of $dx/dt = Ax$.

Problem Q2S11

FIND ALL ASYMPTOTICALLY STABLE EQUILIBRIA OF THE STATE SPACE MODEL

$$\dot{x}(t) = x(t)[|x(t)|^2 - 1] \quad (x(t) \in \mathbb{R}^3).$$

**Hint:** to deal with the non-zero equilibria, note that an asymptotically stable equilibrium cannot be a limit of other equilibria.
Problem Q2S12

For every equilibrium of the state space model \( x(t + 1) = f(x(t)) \), where \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is defined by

\[
f \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1x_2 \\ x_2x_3 \\ x_1x_3 \end{bmatrix}
\]

Determine whether it is asymptotically stable or not stable.

Hint: this is a direct application of the indirect Lyapunov’s method\(^1\).

Problem Q2S13

DT SISO system \( S \) with input \( w = w(t) \) and output \( e = e(t) \) is defined by

\[
e(t) = \min_{0 \leq \tau \leq t} |w(t)|.
\]

Find the L2 gain of \( S \). Explain your answer.

Hint: \(|e(t)| \leq |w(t)|\) for all input/output pairs, and \( e(t) \equiv 1 \) when \( w(t) \equiv 1 \).

Problem Q2S14

CT SISO system \( S \) with input \( w = w(t) \) and output \( e = e(t) \) is defined by \( e(t) = \sin(w(t)/2) \). Find the L2 gain of \( S \). Explain your answer.

Hint: recall how L2 gains of systems \( w(t) \mapsto u(t) = w(t)/2 \) and \( u(t) \mapsto e(t) = \sin(u(t)) \) were computed.

\(^1\)The pun is intended.