Massachusetts Institute of Technology

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6.242, Fall 2004: MODEL REDUCTION [*]

Hankel Optimal Model Reduction[1]

This lecture covers both the theory and an algorithmic side of Hankel optimal model order reduction.

10.1 Basic properties of Hankel operators

This section provides preliminary backgroung for Hankel optimal model reduction.

10.1.1 Hankel operators

Let L^2_r denote the set of all integrable functions $e : \mathbf{R} \mapsto \mathbf{R}^r$ such that

$$\int_{-\infty}^{\infty} |e(t)|^2 dt < \infty.$$

Let $L^2_r(-\infty, 0)$ denote the subset of L^2_r which consist of functions e such that $e(t) = 0$ for $t \geq 0$. The elements of $L^2_r(-\infty, 0)$ will be called *anti-causal* in this lecture. Similarly, let $L^2_r(0, \infty)$ be the subset of functions $e \in L^2_r$ such that $e(t) = 0$ for $t < 0$. The elements of $L^2_r(0, \infty)$ will be called *causal*.

Let $G = G(s)$ be a k-by-m matrix-valued function (not necessarily a rational one), bounded on the $j\omega$-axis. The corresponding *Hankel operator* $\mathcal{H} = \mathcal{H}_G$ is the linear transformation which maps anti-causal square integrable functions $f \in L^2_r(-\infty, 0)$ to

[1] Version of November 10, 2004

causal square integrable functions $h = \mathcal{H}_G f \in L_r^2(0,\infty)$ according to the following rule: $h(t) = y(t)u(t)$, where $y(t)$ is the inverse Fourier transform of $Y(jw) = G(j\omega)F(j\omega)$, $F(j\omega)$ is the Fourier transform of $f(t)$, and $u(t)$ is the unit step function

$$u(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

In terms of the (stable, but not necessarily causal) LTI system defined by G, the Hankel operator maps anti-causal inputs $f = f(t)$ to the causal parts $h = h(t) = y(t)u(t)$ of the complete system response $y(t)$. In particular, when G is anti-stable, i.e. is a proper rational transfer matrix without poles s with $\mathrm{Re}(s) \leq 0$, the associated LTI system is anti-causal, and hence the resulting Hankel operator $\mathcal{H}_G$ is zero. More generally, adding an anti-stable component to G does not affect the resulting $\mathcal{G}$.

10.1.2 Hankel matrices

Let $a > 0$ be a fixed positive number. Then functions

$$\Theta_k(j\omega) = \frac{\sqrt{2a}}{s+a}\left(\frac{a-s}{a+s}\right)^k, \quad k = 0, 1, 2, \ldots$$

form an orthonormal basis in the space of stable strictly proper transfer functions, in the sense that for every such function $H = H(s)$ there exists a square summable sequence of real numbers $h_0, h_1, h_2, \ldots$ satisfying

$$H(j\omega) = \sum_{k=0}^{\infty} h_k \Theta_k(j\omega),$$

in the sense that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|H(j\omega) - \sum_{k=0}^{N} h_k \Theta_k(j\omega)\right|^2 d\omega = \sum_{k=N+1}^{\infty} |h_k|^2 \to 0 \text{ as } N \to \infty.$$

In a similar sense, the inverse Fourier transforms $\theta_k = \theta_k(t)$ of $\Theta_k = \Theta_k(j\omega)$, form an orthonormal basis in $L_1^2(0,\infty)$, and the inverse Fourier transforms $\theta_k(-t)$ of $\Theta_k(-j\omega)$ form an orthonormal basis in $L_1^2(0,\infty)$.

The following lemma allows one to establish a matrix representation of a Hankel operator with respect to input basis $\{\theta_k(-t)\}_{k=0}^{\infty}$ and output basis $\{\theta_k(t)\}_{k=0}^{\infty}$.

Lemma 10.1 *Let*

$$g_k = \frac{1}{\pi}\int_{-\infty}^{\infty} G(j\omega)\left(\frac{a+j\omega}{a-j\omega}\right)^k \frac{ad\omega}{a^2+\omega^2}.$$

Then the result $h = h(t)$ of applying $\mathcal{H}_G$ to $f(t) = \theta_r(-t)$ is given by

$$h(t) = \sum_{k=0}^{\infty} g_{r+k+1}\theta_k(t).$$

An important implication of the lemma is that the matrix of $\mathcal{H}_G$ with respect to the input/output bases $\{\theta_k(-t)\}_{k=0}^{\infty}$, $\{\theta_k(t)\}_{k=0}^{\infty}$ is the *Hankel matrix*

$$\Gamma_G = \begin{bmatrix} g_1 & g_2 & g_3 & g_4 & \\ g_2 & g_3 & g_4 & & \\ g_3 & g_4 & & & \\ g_4 & & & & \\ & & & & \ddots \end{bmatrix}.$$

In general, g_k are matrices with real coefficients, in which case Γ_G is called the *block Hankel matrix.*

Proof Consider the decomposition

$$h(t) = \sum_{k=0}^{\infty} h_k\theta_k(t).$$

By orthonormality of $\theta_k(\cdot)$,

$$h_k = \int_0^{\infty} h(t)\theta_k(t)dt = \int_{-\infty}^{\infty} y(t)\theta_k(t)dt,$$

where y is the response of the stable LTI system associated with G to $f(t) = \theta_r(-t)$. By the Parceval formula,

$$h_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta_k(-j\omega)G(j\omega)\Theta_r(-j\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)\left(\frac{a+j\omega}{a-j\omega}\right)^{k+r+1} \frac{2ad\omega}{(a+j\omega)(a-j\omega)} = g_{k+r+1}.$$

∎

10.1.3 Singular values of a Hankel operator

Let M be an a-by-b matrix representing a linear transformation from $\mathbf{R}^b$ to $\mathbf{R}^a$. Remember that the *operator norm* of M is defined as the minimal upper bound for all ratios $|Mv|/|v|$, where v ranges over the set of all non-zero vectors in $\mathbf{R}^b$. In addition, the r-th singular number of M can be defined as the minimal operator norm of the difference $\Delta = M - \hat{M}$, where $\hat{M}$ ranges over the set of all matrices with rank less than r.

These definitions extend naturally to linear transformations of other normed vector spaces (possibly infinite dimensional). In particular, for a linear transformation M from $L^2_m(-\infty)$ to $L^2_k(0,\infty)$, its *operator norm* is defined as the square root of the minimal upper bound for the ratio

$$\int_0^\infty |(Mf)(t)|^2 dt \Big/ \int_{-\infty}^0 |f(t)|^2 dt,$$

where

$$\int_{-\infty}^0 |f(t)|^2 dt > 0.$$

Such transformation M is said to have *rank* less than r if for every family of r functions $f_1, \dots, f_r \in L^2_k(-\infty, 0)$ there exist constants $c_1, \dots, c_r$, not all equal to zero, such that

$$c_1(Mf_1) + \dots + c_r(Mf_r) \equiv 0.$$

Finally, the r-th singular number of M can be defined as the minimal operator norm of the difference $\Delta = M - \hat{M}$, where $\hat{M}$ ranges over the set of all matrices with rank less than r.

This allows us to talk about the k-th singular number of the Hankel operator $\mathcal{H}_G$ associated with a given matrix-valued function $G = G(j\omega)$, bounded on the imaginary axis. The largest singular number is called the *Hankel norm* $\|G\|_H$ of G, while the k-th singular number is called the k-th *Hankel singular number* of G.

For rational transfer matrices G, calculation of singular numbers of the corresponding Hankel operator can be done using observability and controllability Gramians. The following theorem was, essentially, proven in the lectures on balanced truncation.

Theorem 10.1 *Let A be an n-by-n Hurwitz matrix, B, C be matrices of dimensions n-by-m and k-by-n respectively, such that the pair (A, B) is controllable, and the pair (C, A) is observable. Let W_c, W_o be the corresponding controllability and observability Gramians. Then, for $G(s) = C(sI - A)^{-1}B$, the Hankel operator $\mathcal{H}_G$ has exactly n positive singular numbers, which are the square roots of the eigenvalues of W_cW_o.*

It is also true that a Hankel operator with a finite number of positive singular numbers is defined by a rational transfer matrix.

Theorem 10.2 *Let $G = G(j\omega)$ be a bounded matrix-valued function defined on the imaginary axis. Let $a > 0$ be a positive number. If the Hankel operator $\mathcal{H}_G$ less than r positive singular numbers, then the coefficients*

$$g_k = \frac{1}{\pi}\int_{-\infty}^{\infty} G(j\omega)\left(\frac{a+j\omega}{a-j\omega}\right)^k \frac{a d\omega}{a^2+\omega^2}$$

coincide for $k > 0$ with such coefficients of a stable strictly proper transfer matrix G_1 of order less than r.

For some non-rational transfer matrices, analytical calculation of σ_i may be possible. For example, the i-th largest singular number of $\mathcal{H}_G$, where $G(s) = \exp(-s)$, equals 1 for all positive i.

In general, singular numbers of $\mathcal{H}_G$ will converge to zero if $G = G(j\omega)$ is continuous on the extended imaginary axis (note that $G(s) = \exp(-s)$ is not continuous at $\omega = \infty$). The converse statement is not true.

10.1.4 The Hankel optimal model reduction setup

Let $G = G(s)$ be a matrix-valued function bounded on the $j\omega$-axis. The task of Hankel optimal model reduction of G calls for finding a stable LTI system $\hat{G}$ of order less than a given positive integer m, such that the Hankel norm $\|\Delta\|_H$ of the difference $\Delta = G - \hat{G}$ is minimal.

Since Hankel operator $\mathcal{H}_G$ represents a "part" of the total LTI system with transfer matrix G, Hankel norm is never larger than H-Infinity norm. Hence, Hankel optimal model reduction setup can be viewed as a *relaxation* of the "original" (H-Infinity optimal) model reduction formulation. While no acceptable solution is available for the H-infinity case, Hankel optimal model reduction has an elegant and algorithmically efficient solution.

10.2 The AAK theorem

The solution of the Hankel optimal model reduction problem is based on the famous Adamyan-Arov-Krein (AAK) theorem, presented in this section.

10.2.1 The AAK Theorem

The famous Adamyan-Arov-Krein theorem provides both a theoretical insight and (taking a constructive proof into account) an explicit algorithm for finding Hankel optimal reduced models.

Theorem 10.3 *Let $G = G(s)$ be a matrix-valued function bounded on the $j\omega$-axis. Let $\sigma_1 \geq \sigma_2 \geq \dots \sigma_m \geq 0$ be the m largest singular values of $\mathcal{H}_G$. Then σ_m is the minimum of $\|G - \hat{G}\|_H$ over the set of all stable systems $\hat{G}$ of order less than m.*

In other words, approximating Hankel operators by general linear transformations of rank less than m cannot be done better (in terms of the minimal L2 gain of the error) than approximating it by Hankel operators of rank less than m.

The proof of the theorem, to be given in this section for the case of a rational transfer matrix $G = G(s)$, is constructive, and provides a simple state space algorithm for calculating the Hankel optimal reduced model.

10.2.2 H-Infinity quality of Hankel optimal reduced models

It is well established by numerical experiments that Hankel optimal reduced models usually offer very high H-Infinity quality of model reduction. A somewhat conservative description of this effect is given by the following extension of the AAK theorem.

Theorem 10.4 *Let $G = G(s)$ be a stable rational function. Assume that the Hankel singular numbers $\sigma_k = \sigma_k(G)$ of G satisfy*

$$\sigma_{m-1} > \sigma_m = \sigma_{m+1} = \dots = \sigma_{m+r-1} > \sigma_{m+r}.$$

$\sigma_k(G) = \sigma_m(G)$ for $m \leq k < m+r$, and $\sigma_{m+r}(G) < \sigma_m(G)$. Let $\tilde{\sigma}_m > \tilde{\sigma}_{m+1} > \tilde{\sigma}_{m+2} > \dots$ be the ordered sequence of different *Hankel singular values of G, starting with $\tilde{\sigma}_m = \sigma_m$ and $\tilde{\sigma}_{m+1} = \sigma_{m+r}$. Then*

(a) there exists a Hankel optimal reduced model $\hat{G}_m^H$ of order less than m such that

$$\|G - \hat{G}_m^H\|_\infty \leq \sigma_m + \sum_{k>m}^{r} \sigma_k;$$

(b) there exists a model $\hat{G}_m^*$ of order less than m such that

$$\|G - \hat{G}_m^*\|_\infty \leq \sum_{k \geq m} \tilde{\sigma}_k.$$

Just as in the case of the basic AAK theorem, the proof of Theorem 10.4 is constructive, and hence provides an explicit algorithm for calculation of reduced models with the described properties. In practice, the actual H-Infinity norm of model reduction error is much smaller.

It is important to remember that the Hankel optimal reduced model is never unique (at least, the "D" terms do not have any effect on the Hankel norm, and hence can be modified arbitrarily). The proven H-Infinity model reduction error bound is guaranteed only for a specially selected Hankel optimal reduced model. Also, the reduced model from (b) is not necessarily a Hankel optimal reduced model.

10.2.3 AAK theorem: general comments on the proof

It is sufficient to consider the case when the dimension of $f = f(t)$ equals the dimension of $y = y(t)$ (otherwise, add zero columns to B or zero rows to C).

Since Hankel operator of an anti-stable system is zero, and rank of a Hankel operator of a system of order less than m is less than m, the inequality

$$\|G - \hat{G}_m\|_H \geq \sigma_m(G)$$

holds when the order of $\hat{G}_m$ is less than m.

What remains to be proven is the existence of a $\hat{G}_m$ of order less than m such that

$$\|G - \hat{G}_m\|_H \geq \sigma_m(G).$$

This will be done by constructing explicitly a state space model of transfer matrix $L(s) = \hat{G}_m^H(s) + F^H(s)$, where $\hat{G}_m^H$ is stable and has order less than m, F^H is anti-stable, and $\|G - L\|_\infty = \sigma_m(G)$. Then, by definition, $\|G - \hat{G}_m^H\|_H \leq \sigma_m(G)$.

Actually, a stronger conclusion will be reached: L can be chosen in such way that

$$E(j\omega)'E(j\omega) = \sigma_m(G)I \quad \forall\ \omega \in \mathbf{R}. \tag{10.1}$$

Condition (10.1) will be used later to derive upper bounds for $\|G - \hat{G}_m^H\|_\infty$.

10.2.4 Partitions of the coefficient matrices

Assume that G is defined by a minimal (controllable and observable) *balanced* finite dimensional state space model

$$\dot{x} = Ax + Bf, \ \ y = Cx \tag{10.2}$$

with a Hurwitz n-by-n matrix A. Without loss of generality, consider the case when the controllability and observability Gramian $W = W_c = W_o$ of (10.2) has the form

$$W = \begin{bmatrix} \Sigma & 0 \\ 0 & \gamma I_r \end{bmatrix},$$

where $\gamma = \sigma_m(G)$, the m-th singular number of G (multiplicity r), is *not* an eigenvalue of $\Sigma = \Sigma' > 0$.

Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

be the corresponding block partitions of A, B, C (for example, A_{22} is an r-by-r matrix). Since

$$AW + WA' = -BB', \quad WA + A'W = -C'C,$$

the blocks A_{ij}, B_i, C_i satisfy the relations

$$\begin{aligned} \Sigma A_{11} + A'_{11}\Sigma &= -C'_1 C_1, &(10.3)\\ \Sigma A_{12} + \gamma A'_{21} &= -C'_1 C_2, &(10.4)\\ \gamma(A_{22} + A'_{22}) &= -C'_2 C_2, &(10.5)\\ A_{11}\Sigma + \Sigma A'_{11} &= -B_1 B'_1, &(10.6)\\ \gamma A_{12} + \Sigma A'_{21} &= -B_1 B'_2, &(10.7)\\ \gamma(A_{22} + A'_{22}) &= -B_2 B'_2. &(10.8) \end{aligned}$$

Let

$$\Delta = \Sigma - \gamma^2 \Sigma^{-1}.$$

Combining (10.3) with (10.6) (multiplied by $\gamma\Sigma^{-1}$ on both sides) yields

$$\Delta A_{11} + A_{11}\Delta = \gamma^2 \Sigma^{-1} B_1 B'_1 \Sigma^{-1} - C'_1 C_1. \tag{10.9}$$

Similarly, combining (10.4) with (10.7) yields

$$A'_{12}\Delta = -C'_2 C_1 + \gamma B_2 B'_1 \Sigma^{-1}. \tag{10.10}$$

Finally, (10.5) together with (10.8) implies that $B_2 B'_2 = C'_2 C_2$, which in turn means that $C'_2 U = B_2$ for some unitary matrix U.

In the following section, it will be useful to know that A_{11} has no eigenvalues on the imaginary axis (though in general it may have eigenvalues with positive and negative real parts).

Lemma 10.2 *Let*

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad c = \begin{bmatrix} c_1 & c_2 \end{bmatrix}, \quad p = p' = \begin{bmatrix} q & 0 \\ 0 & \gamma I \end{bmatrix}$$

be such that

$$pa + a'p = -c'c, \quad ap + pa' = -bb'.$$

If γ^2 is not an eigenvalue of q and a has no eigenvalues on the imaginary axis then a_{11} has no eigenvalues on the imaginary axis.

Proof Assume to the contrary that $a_{11}f = j\omega f$ for some $f \neq 0$. Then

$$-|c_1 f|^2 = -f'c_1'c_1 f = f'(qa_{11} + a_{11}'q)f = j\omega f'qf - j\omega f'qf = 0.$$

Hence $c_1 f = 0$.

$$a_{11}'qf = (-c_1'c_1 - qa_{11})f = -j\omega qf.$$

Then

$$-|b_1'qf|^2 = -(qf)'b_1 b_1'(qf) = (qf)'(a_{11}q + qa_{11}')(qf) = 0.$$

Hence $b_1'qf = 0$. Therefore

$$a_{11}q^2 f = (-b_1 b_1' - qa_{11}')qf = j\omega q^2 f,$$

which, combined with $a_{11}f = j\omega f$, yields

$$a_{11}(q^2 - \gamma^2 I)f = j\omega(q^2 - \gamma^2 I)f. \tag{10.11}$$

On the other hand, equalities

$$a_{12}'q + \gamma a_{21} = -c_2'c_1, \quad a_{21}q + \gamma a_{12}' = -b_2 b_1'$$

imply

$$a_{21}(q^2 - \gamma^2)f = 0. \tag{10.12}$$

Combining (10.11) and (10.12) yields

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} (q^2 - \gamma^2)f \\ 0 \end{bmatrix} = j\omega \begin{bmatrix} (q^2 - \gamma^2)f \\ 0 \end{bmatrix},$$

which contradicts the assumptions. ∎

10.2.5 AAK theorem proof: explicit formulae and certificates

In terms of the matrices introduced in the previous subsection, it is easy to define explicitly a state space model of L, as well as the certificates of the H-Infinity norm bounds for $G-L$.

Lemma 10.3 *Let*

$$\begin{aligned} A_L &= A_{11} - \Delta^{-1}(\gamma^2\Sigma^{-1}B_1 - \gamma C_1'U)B_1'\Sigma^{-1}, && (10.13)\\ B_L &= B_1 + \Delta^{-1}(\gamma^2\Sigma^{-1}B_1 - \gamma C_1'U), && (10.14)\\ C_L &= C_1 - \gamma U B_1'\Sigma^{-1}, && (10.15)\\ D_L &= \gamma U. && (10.16)\end{aligned}$$

Then

(a) *the pair (A_L, B_L) is controllable;*

(b) *the pair (C_L, A_L) is observable;*

(c) *A_L (dimension $n-r$) has $m-1$ eigenvalues with negative real part and $n-r-m+1$ eigenvalues with positive real part;*

(d) *transfer matrix $E(s) = \gamma^{-1}(G(s) - L(s))$ satisfies*

$$E(j\omega)'E(j\omega) = \gamma^2 I.$$

(e) *The identity*

$$\gamma^2|f|^2 - |Cx - C_L x_L - D_L f|^2 - 2\mathrm{Re}\left\{\begin{bmatrix} x \\ x_L \end{bmatrix}' H \begin{bmatrix} Ax + Bf \\ A_L x_L + B_L f \end{bmatrix}\right\} = 0 \ \forall\ x, f, x_L \qquad (10.17)$$

holds for

$$H = H' = \begin{bmatrix} \Sigma & 0 & -\Delta \\ 0 & \gamma I_r & 0 \\ -\Delta & 0 & \Delta \end{bmatrix}.$$

Proof Identity (10.17) in (e) can be checked "by inspection".

Statement (d) follows from (e) by substituting x, x_L, f such that

$$j\omega x = Ax + Bf, \quad j\omega x_L = A_L x_L + B_L f,$$

since the real part equals zero in this case.

To prove (a),(b), note first that, according to (10.17), identities

$$Ha + a'H = -c'c, \quad aH^{-1} + H^{-1}a' = -bb',$$

where

$$H^{-1} = \begin{bmatrix} \gamma^{-2}\Sigma & 0 & \gamma^{-2}\Sigma \\ 0 & \gamma^{-1}I & 0 \\ \gamma^{-2}\Sigma & 0 & \gamma^{-2}\Sigma\Delta^{-1}\Sigma \end{bmatrix},$$

hold for

$$a = \begin{bmatrix} A & 0 \\ 0 & A_L \end{bmatrix}, \quad b = \begin{bmatrix} B \\ B_L \end{bmatrix}, \quad c = \begin{bmatrix} C & -C_L \end{bmatrix}.$$

Hence

$$\Delta A_L + A_L'\Delta = -C_L'C_L, \quad A_L(\Sigma\Delta^{-1}\Sigma) + (\Sigma\Delta^{-1}\Sigma)A_L' = -B_LB_L'.$$

Since Δ and $\Sigma\Delta^{-1}\Sigma$ are not singular, controllability of (A_L, B_L) and observability of (C_L, A_L) will follow if A_L has no eigenvalues on the imaginary axis. However, if $f'A_L = j\omega f'$ for some $f \neq 0$ then $f'B_L' = 0$. Since

$$A_L + B_LB_1'\Sigma^{-1} = A_{11} + B_1B_1'\Sigma^{-1},$$

this would imply

$$f'(A_{11} + B_1B_1'\Sigma^{-1}) = j\omega f'.$$

Since

$$(A_{11} + B_1B_1'\Sigma^{-1})\Sigma + \Sigma(A_{11} + B_1B_1'\Sigma^{-1})' = B_1B_1',$$

this implies $f'B_1 = 0$ and hence $f'A_{11} = j\omega f'$, which is impossible due to Lemma 10.2.

Finally, since (A_L, B_L) is controllable, (C_L, A_L) is observable, and A_L has no eigenvalues on the imaginary axis, statement (c) follows. ■