Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.242, Fall 2004: MODEL REDUCTION *

Reduction of uncertain and parameterized models[1]

This lecture generalizes the classical balanced truncation algorithm and the standard upper bound for its approximation error for models represented as feedback interconnections of known LTI systems and uncertainty blocks described by incremental Integral Quadratic Constraints (IQC).

11.1 Analysis of uncertain models

This section introduces a class of uncertain models, as well as a specific technique for the analysis of such systems.

11.1.1 Static IQC models

Uncertain models considered in this lecture are defined by the following list of parameters:

(a) a real $n_x + n_v + n_y$-by-$n_x + n_w + n_f$ matrix

$$M = \begin{bmatrix} a & b_w & b_f \\ c_v & d_{vw} & d_{vf} \\ c_y & d_{yw} & d_{yf} \end{bmatrix},$$

where a is n_x-by-n_x and d_{vw} is n_v-by-n_w;

[1]Version of November 29, 2004

(b) a decomposition

$$n_x = n_{x1} + n_{x2} + \cdots + n_{xr}$$

of n_x as a sum of $r \geq 1$ positive integers;

(c) a set $\tilde{\sigma} = \{\sigma\}$ of quadratic forms $\sigma = \sigma(\bar{v}, \bar{w})$, where the real vector arguments $\bar{v}, \bar{w}$ have dimensions n_v and n_w respectively, such that

$$\sigma(\bar{v}, 0) \geq 0 \quad \forall\, \bar{v} \in \mathbf{R}^{n_v}.$$

These parameters define a set of feedback systems with n_f-dimensional input f and n_y-dimensional output y, each system defined by the system of equations

$$\begin{bmatrix} z(t) \\ v(t) \\ w(t) \end{bmatrix} = M \begin{bmatrix} x(t) \\ w(t) \\ f(t) \end{bmatrix} \quad \forall\, t, \tag{11.1}$$

$$z_i(\cdot) = \delta_i(x_i(\cdot)) \quad (i = 1, \dots, r), \tag{11.2}$$

$$w(\cdot) = \Delta(v(\cdot)), \tag{11.3}$$

where $z_i(t), x_i(t)$ are the n_{xi}-dimensional components of

$$z(t) = \begin{bmatrix} z_1(t) \\ \vdots \\ z_r(t) \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_r(t) \end{bmatrix},$$

Δ is a causal L2 stable system with n_v-dimensional input and n_w-dimensional output, and δ_i is a linear causal L2 stable SISO system (applied component-wise in (11.2)). Here Δ ranges over the set of all stable causal systems which are *odd* (i.e. $\Delta(-v) = -\Delta(v)$ for all inputs v) and satisfy the *incremental Integral Quadratic Constraints* (abbreviation *IQC*) defined by $\sigma \in \tilde{\sigma}$, in the sense that

$$\int_{-\infty}^{T} \sigma(v_1(t) - v_2(t), w_1(t) - w_2(t))dt \geq 0 \tag{11.4}$$

for all $\sigma \in \tilde{\sigma}$, $w_1(\cdot) = \Delta(v_1(\cdot))$, $w_2(\cdot) = \Delta(v_2(\cdot))$. For $i = 1$, δ_i is the LTI system with transfer function $1/(s+1)$. For $i > 1$, δ_i ranges over the set of all linear causal systems satisfying the condition

$$\int_{-\infty}^{T} \{z_*(t)x_i(t) - |z_*(t)|^2\}dt \geq 0 \tag{11.5}$$

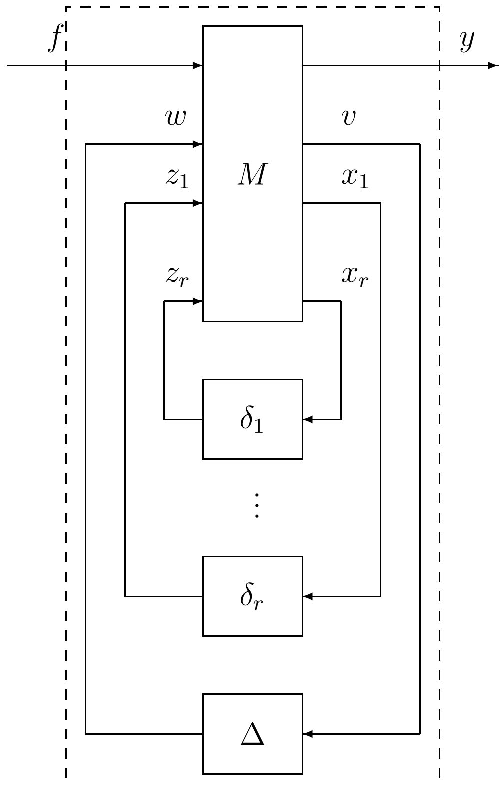

Figure 11.1: A general uncertain model

whenever $z_*(\cdot) = \delta_i(x_*(\cdot))$ is a response of δ_i to an input $x_* = x_*(t)$ of finite energy.

On Figure 11.1, M represents a set of known static linear relations, Δ is the *unstructured* feedback, not to be modified in the reduced model, and δ_i are the *structured* feedbacks, dimensions of which is to be modified in the reduced model.

A very important special case of setup (11.1)-(11.3) is given by system

$$\dot{x}(t) = Ax(t) + B_w w(t) + B_f f(t), \tag{11.6}$$

$$v(t) = C_v x(t) + D_{vw} w(t) + D_{vf} f(t), \tag{11.7}$$

$$y(t) = C_y x(t) + D_{yw} w(t) + D_{yf} f(t), \tag{11.8}$$

$$w(\cdot) = \Delta(v(\cdot)), \tag{11.9}$$

where Δ is defined the same way as before.

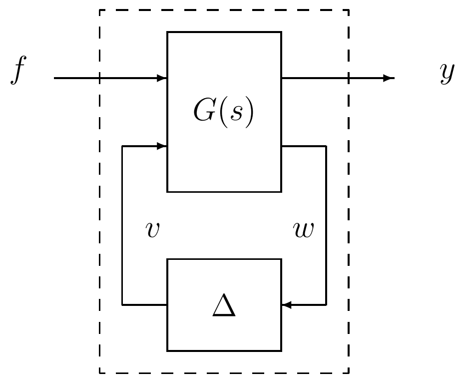

Figure 11.2: Special uncertain model format

Since (11.6) is equivalent to

$$\dot{x}(t) + x(t) = x(t) + Ax(t) + B_w w(t) + B_f f(t),$$

which in turn is identical to

$$x(\cdot) = (s+1)^{-1}(x(\cdot) + Ax(\cdot) + B_w w(t) + B_f f(t)),$$

equations (11.6)-(11.8) can be re-written as (11.1)-(11.3), where $r = 1$, $a = A+I$, $b_w = B_w$, $b_f = B_f$, etc.

Linear time invariant models with no uncertainty are also a special case of (11.1)-(11.3).

11.1.2 Models satisfying static IQC

A stable causal LTI system with transfer function $\delta = \delta(s)$ satisfies the IQC

$$\int_{-\infty}^{T} \{z(t)x(t) - |z(t)|^2\} dt \geq 0$$

for all input/output pairs $(x(t), z(t))$ of finite energy if and only if

$$\text{Re}(\delta(j\omega)) - |\delta(j\omega)|^2 \geq 0 \quad \forall\, \omega \in \mathbf{R},$$

i.e. if and only if the frequency response of δ is contained within the disc of radius 0.5 centered at $s_0 = 0.5$. In particular, transfer function $\delta_1(s) = 1/(s+1)$ and multiplication by a real constant $\delta \in [-1, 1]$ satisfy the IQC.

As long as nonlinear relations are concerned, the transformation $v \mapsto w$ defined by $w(t) = \phi(v(t))$, where $\phi : \mathbf{R} \mapsto \mathbf{R}$ is an odd function such that $\dot{\phi}(\bar{v}) \in [a, b]$, satisfies the incremental IQC

$$\int_{-\infty}^{T} \sigma(v_1(t) - v_2(t), w_1(t) - w_2(t))dt \geq 0$$

for

$$\sigma(\bar{v}, \bar{w}) = (\bar{w} - a\bar{v})(b\bar{v} - \bar{w}).$$

In practice, IQC models represent a conservative bounding of a given nonlinear or time-varying feedback, allowing a relatively simple derivation of upper bounds of closed loop L2 gains (which includes L2 gains of model reduction error systems).

11.1.3 L2 gain bounds for systems with static IQC

Since (11.2),(11.3) imply that

$$\int_{-\infty}^{T} (x_i(t) - z_i(t))' H_i z_i(t) dt \geq 0$$

for an arbitrary family of symmetric matrices $H_i = H_i' \geq 0$, and

$$\int_{-\infty}^{T} \sigma(v(t), w(t))dt \geq 0$$

for all $\sigma \in \tilde{\sigma}$, a sufficient condition for the L2 gain of (11.1)-(11.3) not to exceed $\gamma > 0$ is given by the existence of symmetric matrices $H_i = H_i' \geq 0$ and $\sigma \in \tilde{\sigma}$ such that the quadratic form

$$\gamma^2 |\bar{f}|^2 - |\bar{y}|^2 + \sum_{i=1}^{r} 2(\bar{x}_i - \bar{z}_i)' H_i \bar{z}_i + \sigma(\bar{v}, \bar{w}) \geq 0 \qquad (11.10)$$

is positive semidefinite for all real vectors $\bar{f}$, $\bar{y}$, $\bar{x}_i$, $\bar{z}_i$, $\bar{v}$, $\bar{w}$ of appropriate dimensions, subject to the linear equalities

$$\bar{y} = c_y \bar{x} + d_{yw} \bar{w} + d_{yf} \bar{f}, \;\; \bar{v} = c_v \bar{x} + d_{vw} \bar{w} + d_{vf} \bar{f}.$$

In a certain limited sense, this condition is also necessary, meaning that if such $H_i = H_i' \geq 0$ and $\sigma \in \tilde{\sigma}$ deo not exist then one can find causal systems δ_i, Δ satisfying all listed conditions, such that L2 gain of the resulting system is larger than γ.

When the system has no uncertainty (i.e. $r = 1$ and $n_w = n_v = 0$), the criterion becomes a standard KYP lemma condition for calculating L2 gain of an LTI system.

11.2 Balanced truncation for static IQC models

In this section, a balanced truncation algorithm is developed for reducing the dimensions n_{xi} of the x_i components of static IQC models (11.1)-(11.3). The resulting reduced model will have the same set $\tilde{\sigma}$. An upper bound for the L2 gain (from f to $y - \hat{y}$) of the difference between the elements of the original and reduced families of systems (defined with the same δ_i, Δ) will be obtained.

11.2.1 The simplified setup

The derivation of balanced reduced model is based on working on one particular component of x_i at a time, while leaving the rest intact. Consequently, it becomes sufficient to consider only the case when $r = 1$, since the components z_k and x_k with $k \neq i$ can be appended to w and v respectively without loss of generality. This leads to a simplified setup, in which the original model (11.1)-(11.3) has the form

$$x = \delta(ax + b_w w + b_f f), \tag{11.11}$$
$$v = c_v x + d_{vw} w + d_{vf} f, \tag{11.12}$$
$$y = c_y x + d_{yw} w + d_{yf} f, \tag{11.13}$$
$$w(\cdot) = \Delta(v(\cdot)), \tag{11.14}$$

where δ in (11.11) is a scalar causal bounded linear transformation, satisfying the IQC (11.5), applied component-wise, and Δ is an odd causal bounded transformation satisfying the incremental IQC (11.4). The reduced model is sought in the form

$$x_r = \delta(\hat{a} x_r + \hat{b}_w w_r + \hat{b}_f f), \tag{11.15}$$
$$v_r = \hat{c}_v x_r + d_{vw} w_r + d_{vf} f, \tag{11.16}$$
$$y_r = \hat{c}_y x_r + d_{yw} w_r + d_{yf} f, \tag{11.17}$$
$$w_r(\cdot) = \Delta(v_r(\cdot)), \tag{11.18}$$

where δ, Δ are the same but the dimension of $x_r(t)$ is smaller than that of $x(t)$. An approximation quality is to be asessed by establishing an upper bound for the L2 gain from f to $y - y_r$, provided that δ and Δ in (11.11),(11.14) are the same as in (11.15),(11.18), and satisfy the prescribed IQC.

11.2.2 Weak Gramians of uncertain systems

Recall that for an ordinary state space model

$$\dot{x} = Ax + Bf, \quad y = Cx$$

with a Hurwitz matrix A, the observability Gramian W_o can be characterized as the smallest matrix $W_o = W_o' \geq 0$ such that the quadratic form

$$2\bar{x}'W_oA\bar{x} + |C\bar{x}|^2 \leq 0$$

is negative semidefinite, which means the upper bound

$$|y(t)|^2 \leq -\frac{d}{dt}x(t)'W_ox(t),$$

subject to system equations with $f = 0$. Similarly, the inverse W_c^{-1} of the controllability Gramian is the smallest positive definite matrix such the quadratic form

$$2\bar{x}'W_c^{-1}(A\bar{x} + B\bar{f}) - |\bar{f}|^2 \leq 0$$

is negative semidefinite, which means the lower bound

$$|f(t)|^2 \geq \frac{d}{dt}x(t)'W_c^{-1}x(t),$$

subject to system equations with an arbitrary f.

Accordingly, symmetric positive definite matrices W_o, W_c are called *weak Gramians* of uncertain system (11.11)-(11.14) if there exist quadratic forms $\sigma_o, \sigma_c \in \tilde{\sigma}$ such that

$$2\bar{x}'W_o(a\bar{x} + b_w\bar{w}) + \sigma_o(\bar{v}, \bar{w}) + |\bar{y}|^2 \leq 0 \tag{11.19}$$

subject to

$$\bar{y} = c_y\bar{x} + d_{yw}\bar{w}, \ \ \bar{v} = c_v\bar{x} + d_{vw}\bar{w},$$

and

$$2\bar{x}'W_c^{-1}(a\bar{x} + b_w\bar{w} + b_f\bar{f}) + \sigma_c(\bar{v}, \bar{w}) - |\bar{f}|^2 \leq 0 \tag{11.20}$$

subject to

$$\bar{y} = c_y\bar{x} + d_{yw}\bar{w} + d_{yf}\bar{f}, \ \ \bar{v} = c_v\bar{x} + d_{vw}\bar{w} + d_{vf}\bar{f}.$$

According to this definition, a system may have many weak Gramians. Since a weak observability Gramian W_o^+ establishes an upper bound of output energy in the absense of external inputs, i.e.

$$\int_0^T |y(t)|^2 dt \leq x(0)'W_ox(0) \ \ \text{for} \ \ f \equiv 0,$$

the smaller weak observability Gramians are more useful. Similarly, since a weak controllability Gramian establishes a lower bound of input energy needed to reach a certain state from zero, in the sense that

$$\int_0^T |f(t)|^2 dt \geq x(T)' W_c^{-1} x(T) \quad \text{for} \quad x(0) = 0,$$

the smaller weak controllability Gramians are more useful.

In the ordinary LTI case, the usual observability and controllability Gramians are minimal weak controllability and observability Gramians, respectively. This is not necessarily true in the general case of static IQC models: the set of weak observabilty Gramians may have no single element which is less or equal than any other weak observability Gramian, and the set of weak controllability Gramians may have no single element which is less or equal than any other weak controllability Gramian. Accordingly, practical calculation of weak Gramians should include some sort of minimization. For example, a minimal observability Gramian W_o of uncertain system (11.11)-(11.14) can be defined as the argument of minimum of trace of W_o^+ subject to (11.19). The corresponding optimization with respect to $W_o > 0$, $\tilde{\sigma}_o \in \tilde{\sigma}$ is convex (in most cases, a semidefinite program). Similarly, a minimal controllability Gramian W_c of (11.11)-(11.14) can be defined as inverse of the argument of maximum of trace of $(W_c^+)^{-1}$ subject to (11.20), which also leads to convex optimization with respect to $(W_c^+)^{-1} > 0$, $\sigma_c \in \tilde{\sigma}$.

11.2.3 Balancing and truncation

Applying a non-singular linear change of state coordinates $x := Sx$ in (11.11)-(11.14) yields an equivalent model, in which the coefficient matrices are updated according to

$$a := S^{-1}aS, \; b_w := S^{-1}b_w, \; b_f = S^{-1}b_f, \; c_v := c_vS, \; c_y := c_yS,$$

and a particular pair of weak controllability and observability Gramians W_c, W_o is updated according to

$$W_o^{new} := S'W_oS, \quad W_c^{new} := S^{-1}W_c(S')^{-1}.$$

As in the case of balancing of ordinary state space models, it is possible to choose S in such a way that $W_o^{new} = W_c^{new}$.

More precisely, let $g_1, \dots, g_n$ be an ordered basis of eigenvectors of W_cW_o, i.e. $W_cW_og_k = \gamma_k^2 g_k$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n > 0$, normalized in such a way that $g_k'W_og_k = \gamma_k$. Then, for

$$S = [g_1, g_2, \dots, g_n],$$

the identity

$$S'W_oS = S^{-1}W_c(S')^{-1} = \begin{bmatrix} \gamma_1 & 0 & & \\ 0 & \gamma_2 & & \\ & 0 & \ddots & 0 \\ \gamma_n & & & \end{bmatrix} \tag{11.21}$$

holds. The pair of weak Gramians W_o^{new}, W_c^{new} is called *balanced* for the uncertain system in this case.

Truncation of the uncertain system with balanced weak Gramians is performed by choosing an index k (typically such that $\gamma_k > \gamma_{k+1}$) and defining the coefficients of the truncated uncertain system by

$$\hat{a} = a_{11},\ \hat{b}_w = b_{w1},\ \hat{b}_f = b_{f1},\ \hat{c}_v = c_{v1},\ \hat{c}_y = c_{y1}, \tag{11.22}$$

$$\hat{d}_{vw} = d_{vw},\ \hat{d}_{vf} = d_{vf},\ \hat{d}_{yw} = d_{yw},\ \hat{d}_{yf} = d_{yf}, \tag{11.23}$$

where

$$S^{-1}aS = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ S^{-1}b_w = \begin{bmatrix} b_{w1} \\ b_{w2} \end{bmatrix},\ S^{-1}b_f = \begin{bmatrix} b_{f1} \\ b_{f2} \end{bmatrix},\ S'c_v' = \begin{bmatrix} c_{v1}' \\ S'c_{v2}' \end{bmatrix},\ c_y' = \begin{bmatrix} c_{y1}' \\ c_{y2}' \end{bmatrix}, \tag{11.24}$$

and the number of rows in $a_{11}, b_{w1}, b_{f1}, c_{v1}', c_{y1}'$ equals k.

The following theorem generalized the classical upper bound of the H-Infinity error in balanced truncation.

Theorem 11.1 *Assume that δ is a causal stable linear system satisfying the IQC in (11.5). Assume Δ is a stable causal odd system satisfying the incremental IQC in (11.4) with quadratic forms $\sigma = \sigma_o$ and $\sigma = \sigma_c$. Let W_c, W_o be positive semidefinite satisfying (11.19), (11.20). Let S be a non-singular matrix satisfying (11.21), where $\gamma_1 \geq \cdots \geq \gamma_n \geq 0$. Let the reduced model (11.15)-(11.18) of system (11.11)-(11.14) be defined by (11.22)-(11.24). Then L2 gain from f to $y - y_r$ does not exceed $2\sum_{i>k}\gamma_i$, where each value of γ_i is counted once.*

11.2.4 Proof of the truncation error bound

The proof is based on the following observation.

Lemma 11.1 *Assume that matrices $P = P'$, $Q = Q'$, a_{ij}, b_{wi}, b_{fi}, c_{vi}, c_{yi}, where $i, j \in \{1, 2\}$, and a scalar $\gamma \geq 0$ satisfy*

$$2x_1'P(z_1 - x_1) + 2\gamma x_2'(z_2 - x_2) + \sigma_o(v, w) + |y|^2 \leq 0, \tag{11.25}$$

subject to

$$\begin{aligned} z_1 &= a_{11}x_1 + a_{12}x_2 + b_{w1}w), \\ z_2 &= a_{21}x_1 + a_{22}x_2 + b_{w2}w, \\ v &= c_{v1}x_1 + c_{v2}x_2 + d_{vw}w, \\ y &= c_{y1}x_1 + c_{y2}x_2 + d_{yw}w, \end{aligned}$$

and

$$2x_1'Q(z_1 - x_1) + 2\gamma^{-1}x_2'(z_2 - x_2) + \sigma_c(v, w) - |f|^2 \le 0, \tag{11.26}$$

subject to

$$\begin{aligned} z_1 &= a_{11}x_1 + a_{12}x_2 + b_{w1}w + b_{f1}f, \\ z_2 &= a_{21}x_1 + a_{22}x_2 + b_{w2}w + b_{f2}f, \\ v &= c_{v1}x_1 + c_{v2}x_2 + d_{vw}w + d_{vf}f, \end{aligned}$$

for all real vectors x_1, x_2, w, f of compatible dimensions. Then

(a) Matrices $P = P'$, $Q = Q'$, a_{11}, b_{w1}, b_{f1}, c_{v1}, c_{y1} satisfy

$$2x_r'P(z_r - x_r) + \sigma_o(v_r, w_r) + |y_r|^2 \le 0, \tag{11.27}$$

subject to

$$\begin{aligned} z_r &= a_{11}x_r + b_{w1}w_r, \\ v_r &= c_{v1}x_r + d_{vw}w_r, \\ y_r &= c_{y1}x_r + d_{yw}w_r, \end{aligned}$$

and

$$2x_r'Q(z_r - x_r) + \sigma_c(v_r, w_r) - |f|^2 \le 0, \tag{11.28}$$

subject to

$$\begin{aligned} z_r &= a_{11}x_r + b_{w1}w_r + b_{f1}f, \\ v_r &= c_{v1}x_r + d_{vw}w_r + d_{vf}f, \end{aligned}$$

for all real vectors x_r, w_r, f of compatible dimensions.

(b) Matrix

$$H=\begin{bmatrix}\gamma^2Q+P & 0 & \gamma^2Q-P\\ 0 & 2\gamma & 0\\ \gamma^2Q-P & 0 & \gamma^2Q+P\end{bmatrix}$$

satisfies

$$|y-y_r|^2-4\gamma^2|f|^2+$$
$$+2\begin{bmatrix}x_1\\ x_2\\ x_r\end{bmatrix}' H\begin{bmatrix}z_1-x_1\\ z_2-x_2\\ z_3-x_3\end{bmatrix}+\sigma_o(v-v_r,w-w_r)+\gamma^2\sigma_c(v+v_r,w+w_r)\le 0, \quad (11.29)$$

subject to

$$\begin{aligned}
z_1 &= a_{11}x_1+a_{12}x_2+b_{w1}w+b_{f1}f,\\
z_2 &= a_{21}x_1+a_{22}x_2+b_{w2}w+b_{f2}f,\\
z_r &= a_{11}x_r+b_{w1}w_r+b_{f1}f,\\
v &= c_{v1}x_1+c_{v2}x_2+d_{vw}w+d_{vf}f,\\
v_r &= c_{v1}x_r+d_{vw}w+d_{vf}f,\\
y &= c_{y1}x_1+c_{y2}x_2+d_{yw}w+d_{yf}f,\\
y_r &= c_{y1}x_r+d_{yw}w+d_{yf}f,
\end{aligned}$$

for all vectors $x1, x_2, x_r, w, w_r, f$ *of compatible dimensions.*

The assumptions of the lemma mean that a couple of weak Gramians of system (11.11)-(11.14) is given by

$$W_o=\begin{bmatrix}P & 0\\ 0 & \gamma I_r\end{bmatrix},\quad W_c^{-1}=\begin{bmatrix}Q & 0\\ 0 & \gamma^{-1}I_r\end{bmatrix}.$$

Here γ plays the role of the smallest of the “singular numbers” γ_i, of multiplicity r. Conclusions (a) and (b) describe properties of the system resulting from truncating the last r states. Condition (a), which follows from the assumptions by substituting $x_2=0$, means that P and Q are weak observability Gramians for the reduced system. Condition (b), obtained by adding the inequality from (11.25) (with x_1 replaced by x_1-x_r and w replaced by $w-w_r$) to the inequality from (11.26) (with x_1 replaced by x_1+x_r, w replaced by $w+w_r$, and f replaced by $2f$), multiplied by γ^2, establishes an L2 gain bound of 2γ from f to the error system output $y-y_r$. Together, (a) and (b) yield a proof of Theorem 11.1 by induction.