Massachusetts Institute of Technology

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6.242, Fall 2004: MODEL REDUCTION *

Reduction of systems with real parameters[1]

In this lecture, model reduction of LTI systems depending on real parameters is discussed. While it appears that there is no single approach offering a satisfactory solution to the problem, a number of techniques studied in the previous lectures can be extended to the case of parameterized models.

To streamline the presentation, we will only consider the case of a single constant a-priori bounded parameter, when the original model is defined by

$$\dot{x}(t) = (A + F\delta L)x(t) + Bf(t), \;\; y(t) = Cx(t), \tag{12.1}$$

where $\delta \in [-1, 1]$ is the parameter, A, B, C, F, L are given matrices of dimensions n-by-n, n-by-m, k-by-n, n-by-d, and d-by-n respectively. It is assumed that $A + F\delta L$ is a Hurwitz matrix for all $\delta \in [-1, 1]$. The objective is to derive a reduced model in the form

$$\dot{x}_r(t) = A_r(\delta)x_r(\delta) + B_r(\delta)f(t), \quad y_r(t) = C_r(\delta)x_r(t), \tag{12.2}$$

where $A_r(\delta), B_r(\delta), C_r(\delta)$ are easy to calculate for a given $\delta \in [-1, 1]$. The reduced model in (12.2) is acceptable when, for every $\delta \in [-1, 1]$, $A_r(\delta)$ is a Hurwitz matrix and the H-Infinity norm of the model reduction error system is small.

[1] Version of December 1, 2004

12.1 Treating parameter dependence as model uncertainty

System (12.1) can be viewed as an uncertain system

$$\dot{x} = Ax + Fw + Bf, \ y = Cx, \ v = Lx, \ w(\cdot) = \Delta(v(\cdot)), \tag{12.3}$$

where Δ is the operator of multiplication by constant $\delta \in [-1, 1]$.

As Δ is linear, and can be described by a set of integral quadratic constraints (IQC) the balanced truncation approach can be applied to produce reduced models with guaranteed stability and error bounds.

12.1.1 Static IQC for real parameter uncertainty

Static IQC describe the dependence of w on v in terms of integral relations

$$\int_0^\infty \sigma(x(t), w(t), f(t))dt \geq 0, \tag{12.4}$$

where σ are quadratic forms, which has to be satisfied whenever x, v, w, f are square integrable signals satisfying conditions (12.3). Once a set $\tilde{\sigma}$ of such σ is available, the (weak) controllability and observability Gramians can be defined as symmetric matrix solutions $W_o, W_c > 0$ of the quadratic form inequalities

$$2x'W_o(Ax + Fw) + |Cx|^2 + \sigma_o(x, w, 0) \leq 0 \ \ \forall \ x, w, \tag{12.5}$$

$$2x'W_c^{-1}(Ax + Bf + Fw) - |f|^2 + \sigma_c(x, w, f) \leq 0 \ \ \forall \ x, w, f, \tag{12.6}$$

where $\sigma_o, \sigma_c \in \tilde{\sigma}$, and the standard balanced truncation algorithm utilizing W_o, W_c can be applied to get a reduced projected model with guaranteed stability and the usual error bounds.

A set of IQC for $\Delta(v) = \delta v$, where $\delta \in [-1, 1]$ is an uncertain *constant* is given by

$$\sigma(x, w, f) = \sigma_R^0(x, w, f) = (Lx)'R(Lx) - w'Rw, \quad R = R' \geq 0 \tag{12.7}$$

where $R = R' \geq 0$ is an arbitrary positive semidefinite matrix. Indeed, since $w(t) = \delta v(t)$, where $\delta \in [-1, 1]$ is a scalar, we have

$$v'Rv - w'Rw = (1 - \delta^2)v'Rv \geq 0.$$

In this case, (12.5) is equivalent to the LMI with respect to $W_o, R_0 \geq 0$:

$$\begin{bmatrix} W_oA + A'W_o + C'C + L'R_oL & W_oF \\ F'W_o & -R_o \end{bmatrix} \leq 0,$$

which, for $R_o > 0$, can be transformed into

$$W_oA + A'W_o + C'C + L'R_oL + W_oFR_o^{-1}F'W_o \leq 0.$$

As one can see, the usual Lyapunov inequality (linear with respect to W_o) is transformed into a Riccati inequality (convex quadratic with respect to W_o), parameterized by $R_o = R_o' > 0$.

Similarly, (12.6) is equivalent to the LMI with respect to $W_c^{-1}, R_0 > 0$:

$$\begin{bmatrix} W_c^{-1}A + A'W_c^{-1} + L'R_cL & W_c^{-1}B & W_c^{-1}F \\ B'W_c^{-1} & -I_m & 0 \\ F'W_c^{-1} & 0 & -R_c \end{bmatrix} \leq 0,$$

which can be transformed into the Riccati inequality

$$AW_c + W_cA' + W_cL'R_cLW_c + BB' + FR_c^{-1}F' \leq 0,$$

quadratic convex with respect to W_c and parameterized by $R_c = R_c' > 0$.

It is important to understand that IQC (12.4) defined by (12.7) are satisfied for a very large class of systems Δ. For example, multiplication by a *time-varying* scalar $\delta = \delta(t)$ satisfies the same IQC. Moreover, it van be shown that the IQC are also valid when $\delta = \delta(s)$ is an arbitrary stable SISO LTI system with H-Infinity norm not exceeding 1. Informally, it would be fair to say that IQC (12.4) defined by (12.7) does not utilize time-invariance and memorylessness properties of system Δ defined as multiplication by a constant scalar.

Additional IQC can be obtained by noticing that

$$\int_0^\infty 2\delta v(t)'H\dot{v}(t)dt = \delta v(t)'Hv(t)|_0^\infty = 0$$

when $H = H'$ is a symmetric matrix, x, w, f are square integrable and $x(0) = 0$. In other terms, quadratic forms

$$\sigma(x, w, f) = \sigma_H^1(x, w, f) = 2(Lx)'HL(Ax + Fw + Bf), \quad H = H' \qquad (12.8)$$

define valid static IQC for Δ. Note that quadratic forms σ_H^1 do not define, in general, IQC valid when Δ is time-varying or has memory.

If $LF = 0$ and $LB = 0$, one can use the equality

$$\int_0^\infty \delta v(t)'R\ddot{v}(t)dt = -\delta \int_0^\infty \dot{v}(t)'R\dot{v}(t)dt$$

to derive additional IQC (12.4) with

$$\sigma(x,w,f) = \sigma_R^+(x,w,f) = 2w'RLA(Ax+Fw+Bf) + (LAx)'R(LAx), \quad R = R' \geq 0, \tag{12.9}$$

and

$$\sigma(x,w,f) = \sigma_R^-(x,w,f) = -2w'RLA(Ax+Fw+Bf) + (LAx)'R(LAx), \quad R = R' \geq 0. \tag{12.10}$$

Indeed, these tricks with differentiation can be continued further when $LAB = 0$ and $LAF = 0$.