Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.242, Fall 2004: MODEL REDUCTION *

L2 gains and system approximation quality[1]

This lecture discusses the utility of L2 gains and related measures in assessing quality of system approximation.

3.1 L2 gain and its calculation

L2 gain is a popular measure of smallness for system approximation errors and dynamical perturbations. This section covers main definitions and L2 gain calculation methods.

3.1.1 H-Infinity norm and uniform matching of CT LTI system response

Consider a causal convolution model of an LTI system, possibly of an infinite order, defined by the impulse response matrix $g = g(t)$ or by a transfer matrix $G = G(s)$. Remember that such system defines the output y as a convolution integral

$$y(t) = \int_0^\infty h(\tau)f(t-\tau)d\tau,$$

where f is assumed to vanish fast enough as $t \to -\infty$.

Let $\hat{g} = \hat{g}(t)$ and $\hat{G} = \hat{G}(s)$ be the impulse response and the transfer matrix of another convolution model, intended to serve as a simplified approximation of the original one. How does one quantify the quality of this approximation?

[1]Version of September 19, 2004

One common way of measuring approximation quality is by comparing system responses $y_0 = y_0(t)$ and $\hat{y}_0 = \hat{y}_0(t)$ to a particular "testing" input $f_0 = f_0(t)$, as shown on Figure 3.1.

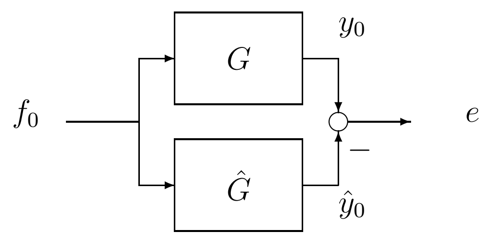

Figure 3.1: Output matching error

In particular, the "Gaussian" $f_0(t) = \exp(-\rho t^2)$ and "unit step"

$$f_0(t) = u(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0, \end{cases}$$

are frequently used as test signals. While comparing time domain responses to special testing inputs is a valuable way of measuring approximation quality, it is usually not easy to justify a particular selection of a test signal, and even more difficult to give a formal proof of benefits of reducing this particular type of approximation error.

In this class, when evaluating quality of reduced LTI models, we will be most interested in a matching *frequency responses* of LTI systems. This leads to the so-called *H-Infinity norm* of the difference $H(s) = G(s) - \hat{G}(s)$ as the approximation error measure of choice.

Definition Let $H = H(s)$ be the Laplace transform of an impulse response matrix $h = h(t)$ such that $h(t) = 0$ for $t < 0$. The *H-Infinity norm* of H, denoted by $\|H\|_\infty$, is given by

$$\|H\|_\infty = \sup_{\mathrm{Re}(s)>0} \|H(s)\|$$

when $H(s)$ is well defined in the open right half plane, and equals $+\infty$ otherwise.

For example, H-Infinity norm of $H(s) = 1/(s+a)$ equals $1/a$ for $a > 0$, and $+\infty$ for $a \leq 0$. H-Infinity norm of $H(s) = \exp(-as)$ equals 1 for $a \geq 0$, and is not defined for $a < 0$ since H is not a transfer function of a causal LTI system in this case.

Note that using an H-Infinity model reduction error measure $\|G - \hat{G}\|_\infty$ automatically requires exact matching of all *unstable* poles, because H-Infinity norm of a transfer matrix with a pole in the closed right half plane equals infinity.

The following theorem (presented without a proof here) shows the relation between the H-Infinity norm and maximal norm of the frequency response.

Theorem 3.1 *Let*

$$h(t) = h_c(t) + \sum_{k=0}^{\infty} h_k \delta(t - \tau_k),$$

where

$$\tau_k \geq 0, \; \sum_{k=0}^{\infty} \|h_k\| < \infty, \; \int_0^{\infty} \|h(t)\| dt < \infty.$$

Then

$$\|H\|_\infty = \sup_{\omega \in \mathbf{R}} \|H(j\omega)\|.$$

As it follows from the theorem, an important interpretation of the H-Infinity norm is that of a maximal amplification the corresponding LTI system can give to a sinusoidal input. An variety of model reduction error measures can be obtained by using *weighted* H-Infinity norms, such as

$$\|W_o(G(s) - \hat{G}(s))W_i(s)\|_\infty,$$

where W_o, W_i are given rational transfer matrices without poles in the closed right half plane.

3.1.2 H-Infinity norm as L2 gain

A very important characteristic of an input/output system is its *L2 gain*.

Definition A system is said to have *finite L2 gain* if there exists a $\gamma > 0$ such that

$$\inf_{t>T} \int_T^t \{|\gamma^2 |f(\tau)|^2 - |y(\tau)|^2\} d\tau > -\infty$$

for every input/output pair (f, y) and every $T \in \mathbf{R}$. The infimum of such $\gamma > 0$ is called the *L2 gain* of the system.

The fact that a system has finite L2 gain serves well as a definition of stability, and the L2 gain itself can be viewed as a measure of “size” of a system.

For example, the memoryless nonlinear system defined by

$$y(t) = \sin(f(t))$$

has L2 gain 1. To prove this, note first that

$$|\sin(\bar{y})| \le |\bar{y}|$$

for all $\bar{y} \in \mathbf{R}$, and hence

$$\int_T^\tau \{|f(t)|^2 - |y(t)|^2\} dt \ge 0$$

for all input/output pairs. This implies that L2 gain is finite and is not larger than 1. On the other hand, for every $\gamma < 1$ there exists $\epsilon > 0$ such that $\sin(\epsilon) > \gamma\epsilon$. Since a constant input $f(t) \equiv \epsilon$ produces a constant output $y(t) \equiv \sin(\epsilon)$, the integral

$$\int_T^\tau \{|f(t)|^2 - |y(t)|^2\} dt$$

converges to $-\infty$ as $\tau \to \infty$.

Another instructive example of L2 gain calculation is for the linear time-varying system defined by

$$y(t) = k(t)f(t), \quad k(t) = e^{-t^2}.$$

It is tempting to say that L2 gain equals 1, but it is actually equal to zero, since the definition of the L2 gain concerns only with the *asymptotic* behavior of $k = k(t)$ as $t \to \infty$.

The following theorems (offered here without a proof) state that, for a causal LTI system, L2 gain equals the H-Infinity norm of the transfer matrix.

Theorem 3.2 *L2 gain of a causal LTI convolution model equals H-Infinity norm of its transfer matrix.*

In the following theorem we consider a state space model

$$dx/dt = Ax + Bf, \quad y = Cx + Df, \quad x(0) \text{ not fixed}, \tag{3.1}$$

in which every initial condition $x(0)$ generates an admissible output, so that one input $f = f(t)$ produces many outputs $y = y(t)$.

Theorem 3.3 *If A is a Hurwitz matrix (i.e. all eigenvalues have negative real part) then L2 gain of system (3.1) equals H-Infinity norm of its transfer matrix*

$$H(s) = D + C(sI - A)^{-1}B.$$

In particular, while formally the state space model

$$dx/dt = -x + f, \quad y = x, \quad x(0) \text{ not fixed},$$

and the impulse response $h(t) = \exp(-t)u(t)$ define two different systems H_1 and H_2 (the second has outputs uniquely defined by the inputs, while the first is multi-valued), L2 gain of the difference $\Delta = H_1 - H_2$ equals zero, which means that the two systems are equivalent for most practical purposes.

3.1.3 Incremental L2 gain

Sometimes L2 gain is too primitive as a measure of model reduction quality. A frequently used alternative is the so-called *incremental L2 gain.*

Definition A system with input f and output y is said to have *finite incremental L2 gain* if there exists a constant $\gamma \geq 0$ such that the inequality

$$\int_{-\infty}^{\infty} \{\gamma^2 |f_1(t) - f_2(t)|^2 - |y_1(t) - y_2(t)|^2\} dt \geq 0$$

holds for all input/output pairs (f_1, y_1) and (f_2, y_2) such that

$$\int_{-\infty}^{\infty} |f_1(t) - f_2(t)|^2 dt < \infty.$$

The minimal $\gamma \geq 0$ satisfying this condition is called the *incremental L2 gain* of the system.

According to the definition, systems with finite incremental gain have a unique output corresponding to each input. It is also easy to see that, for LTI convolution models, the incremental L2 gain equals the usual L2 gain. However, for nonlinear or time-varying systems, these two gains are different. This can be illustrated by the following statement.

Theorem 3.4 *For a given function* $\phi: \mathbf{R} \mapsto \mathbf{R}$,

(a) *L2 gain of system* $f(t) \mapsto y(t) = \phi(f(t))$ *equals*

$$\inf\{\gamma > 0 : \ |\phi(\bar{f})| \leq \gamma |\bar{f}| \quad \forall \ \bar{f} \in \mathbf{R}\};$$

(b) *incremental L2 gain of system* $f(t) \mapsto y(t) = \phi(f(t))$ *equals*

$$\sup_{\bar{f}_1 \neq \bar{f}_2} \frac{|\phi(\bar{f}_1) - \phi(\bar{f}_2)|}{|\bar{f}_1 - \bar{f}_2|};$$

(c) L2 gain of system $f(t) \mapsto y(t) = \phi(t)f(t)$ equals

$$\lim_{T\to+\infty} \sup_{t>T} |\phi(t)|;$$

(d) incremental L2 gain of system $f(t) \mapsto y(t) = \phi(t)f(t)$ equals

$$\sup_{t\in\mathbf{R}} |\phi(t)|.$$

For example, incremental L2 gain of system $f(t) \mapsto f(t)\cos f(t)$ equals infinity, while its L2 gain equals 1. On the other hand, incremental gain of system $f(t) \mapsto \exp(-t^2)f(t)$ equals 1, while its L2 gain equals 0.

3.2 Small gain theorems

Small gain theorems are among the main reasons to minimize model error gains as measures of model reduction quality. The main motivation for the techniques described in this section comes from the following setup.

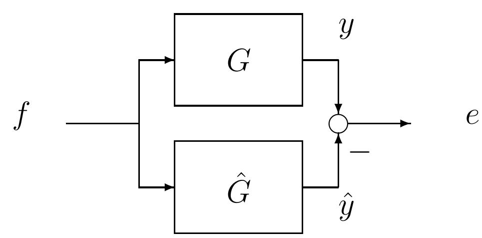

Figure 3.2: Model reduction error system

Let G and $\hat{G}$ be the original system and its reduced model, respectively. Let $\Delta = G - \hat{G}$ be the *model reduction error* system $f \mapsto e$, defined by the block diagram on Figure 3.2. Consider the situation when one has to derive properties of a feedback system which has G as a subsystem (the left diagram on Figure 3.3) from the properties of a similar interconnection in which G is replaced by $\hat{G}$ (the left diagram on Figure 3.3).

Let system H with inputs $w, \bar{e}$ and outputs $\bar{z}, \bar{f}$ be defined by the block diagram on Figure 3.4.

In terms of H and Δ, the system perturbation described on Figure 3.3 has an equivalent description given by Figure 3.5, in which Δ is being replaced by zero.

Two major interpretations of quality of a subsystem approximation are as follows:

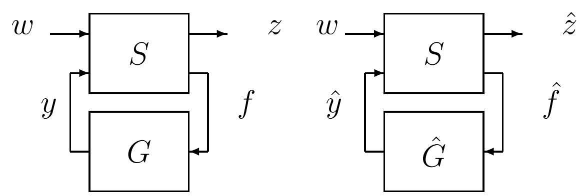

Figure 3.3: Replacing G by $\hat{G}$ in an interconnection

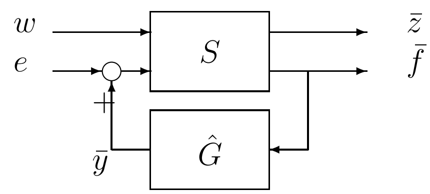

Figure 3.4: System H

(a) A bound on the L2 gain of the full interconnection system (left block diagrams on Figures 3.3 and 3.5) can be derived from a description of H and a gain bound of Δ);

(b) A bound on the energy

$$\int_{-\infty}^{\infty} |z(t) - \hat{z}(t)|^2 dt$$

of the output error can be derived from a description of H and an L2 gain bound of Δ).

3.2.1 Small gain theorem for L2 gain estimation

The following theorem, a version of the famous *small gain condition*, allows one to estimate L2 gain of the full interconnection system (left block diagram on Figure 3.5) by using the information on L2 gains of H and Δ.

Theorem 3.5 *If L2 gains of H and Δ are not larger than 1 then the feedback interconnection on the left block diagram of Figure 3.5 has L2 gain (from w to z) not larger than 1.*

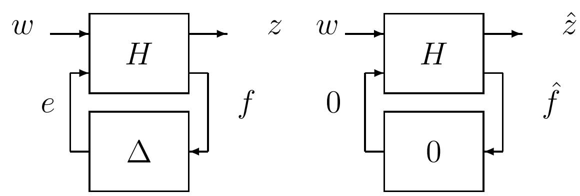

Figure 3.5: Replacing Δ by zero in an interconnection

Proof Consider the signals consistent with the block diagram. By assumption about Δ,

$$\inf_{T>0}\int_0^T\{|f|^2-|e|^2\}dt>-\infty.$$

By assumption about H,

$$\inf_{T>0}\int_0^T\{|w|^2+|e|^2-|z|^2-|f|^2\}dt>-\infty.$$

Adding these two inequalities yields

$$\inf_{T>0}\int_0^T\{|w|^2-|z|^2\}dt>-\infty.$$

■

For simplicity, the L2 small gain theorem is formulated for the case of unit L2 gain bounds. By re-scaling inputs and outputs of H, one obtains a more general condition, which applies to the case of arbitrary L2 bound of Δ and the closed loop L2 bound to be proven.

3.2.2 Output matching error estimation

Let us say that system Δ with input f and output e has *L2 norm* not exceeding $\gamma\geq 0$ if the inequality

$$\int_{-\infty}^{\infty}|y(y)|^2dt\leq\gamma^2\int_{-\infty}^{\infty}|f(t)|^2dt$$

holds for every input/output pair (f,y) of Δ. Note that a system with a finite L2 norm bound must respond with a zero output to a zero input. Informally speaking, L2 norm bound acts as a stronger version of the familiar L2 gain bound, applicable to systems with "zero initial conditions".

In order to estimate the energy

$$\int_{-\infty}^{\infty} |\hat{z}(t) - z(t)|^2 dt$$

of the output matching error $\hat{z} - z$ on Figure 3.5, one can use an incremental L2 gain bound for H, and an L2 norm bound for Δ.

Theorem 3.6 *For the systems defined by Figure 3.5, assume that the incremental L2 gain of H is not larger than 1, and the L2 norm bound of D is not larger than $\gamma \in [0, 1)$. Then*

$$\int_{-\infty}^{\infty} |z(t) - \hat{z}(t)|^2 dt \leq \frac{\gamma^2}{1-\gamma^2} \int_{-\infty}^{\infty} |\hat{f}(t)|^2 dt$$

for every finite energy input w.

Proof Let $\|g\|^2$ denote the energy of signal g. Let signals $w, z, e, f, \hat{z}, \hat{f}$ be consistent with the block diagrams on Figure 3.5. By the incremental L2 gain bound imposed on H,

$$\|z - \hat{z}\|^2 + \|f - \hat{f}\|^2 \leq \|e\|^2.$$

By the L2 norm assumption about Δ,

$$\|e\|^2 \leq \gamma^2 \|f\|^2.$$

By the Cauchy-Schwartz inequality,

$$\gamma^2 \|f\|^2 - \|f - \hat{f}\|^2 \leq \frac{\gamma^2}{1-\gamma^2} \|\hat{f}\|^2,$$

which implies the desired upper bound for $\|\hat{z} - z\|^2$. ■

This theorem can be used to prove upper bounds of system simulation errors caused by replacing a complex subsystem by its reduced model. The assumptions, in particular, mean that the "initial conditions" of the error subsystem Δ are set to "zero" at $t = -\infty$, and the bound is given in terms of the $\hat{f}$ output of the reduced system H.

3.2.3 Small incremental gain theorem

Finding upper bounds for L2 gains of non-linear systems can be quite challenging, even in a low order case. Just as in the case of non-incremental L2 gains, a small gain theorem provides an easy (though not always accurate enough) way for bounding incremental gains of feedback systems.

Theorem 3.7 *For the system defined by the left block diagram on Figure 3.5, the incremental L2 gain is not larger than 1 whenever the incremental L2 gains of H and Δ are not larger than 1.*

The proof of this statement is similar to the non-incremental version described earlier in this lecture.