Massachusetts Institute of Technology

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6.242, Fall 2004: MODEL REDUCTION *

Model reduction by projection: general properties[1]

This lecture covers some basic properties of *projection methods*, a general class of powerful model reduction algorithms.

4.1 Common projection schemes

A projection method can be viewed as application of a "lossy" compression to a system's state, and re-writing equations for the state's dynamics in terms of a compressed representation. Depending on a system and model class (linear vs. non-linear, state-space vs. input/output, time-varying vs. time-invariant, etc.), different implementations of this approach become typical.

4.1.1 Projection of finite order state space LTI models

The sequence of operations for obtaining a reduced CT LTI SS model

$$\frac{d}{dt}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}f(t), \quad \hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}f(t)$$

from the original higher-order model

$$\frac{d}{dt}x(t) = Ax(t) + Bf(t), \quad y(t) = Cx(t) + Df(t)$$

can be described in the following way.

[1] Version of September 19, 2004

Step 1: apply an invertible coordinate change $x(t) = S\bar{x}(t)$, where S is an invertible square matrix, to re-write the original system equations in the form

$$\frac{d}{dt}\bar{x}(t) = \bar{A}\bar{x}(t) + \bar{B}f(t), \quad y(t) = \bar{C}\bar{x}(t) + \bar{D}f(t),$$

where

$$\bar{A} = S^{-1}AS, \quad \bar{B} = S^{-1}B, \quad \bar{C} = CS, \quad \bar{D} = D.$$

Step 2: partition the new state vector $\bar{x}(t)$ as

$$\bar{x}(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix},$$

where the dimension of $\bar{x}_1(t)$ equals the desired order of the reduced system; partition the $\bar{A}, \bar{B}, \bar{C}$ matrices accordingly:

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}.$$

Step 3: define the reduced system by

$$\hat{A} = \bar{A}_{11}, \quad \hat{B} = \bar{B}_1, \quad \hat{C} = \bar{C}_1, \quad \hat{D} = D.$$

The following reasoning stays usually behind such series of manipulations. At *step 1*, the system states are re-arranged to place the "most important" ones as the first few components of the new state vector $\bar{x}(t)$. At *step 2*, one decides which components of $\bar{x}(t)$ are to be "ignored" (set to zero) in the new equations. *Step 3* defines the resulting simplified state equations.

Selecting a "good" initial transformation S, as well as determining how many states of $\bar{x}(t)$ to keep, is what defines a particular projection-based model reduction algorithm. One can invest a lot of effort in finding a good S, and come up with a high quality reduced model. Alternatively, one can obtain S according to some "cheap" strategy, to get a barely adequate reduced model. Actually, it can be shown that, by selecting an arbitrary S, one can obtain an *arbitrary* (subject to some non-essential restrictions) reduced order system from a given model.

For example, an unobservable CT LTI SS model (i.e. the one with the observability matrix M_o not having full rank) can be transformed to a form with

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} \bar{C}_1 & 0 \end{bmatrix}.$$

In this case the reason to consider the state $\bar{x}_2(t)$ "unimportant" is its unobservability, and the projection reduced model

$$\hat{A} = \bar{A}_{11}, \quad \hat{B} = \bar{B}_1, \quad \hat{C} = \bar{C}_1, \quad \hat{D} = D$$

has a transfer matrix which is identical to that of the original system.

As another example, consider the stable causal LTI system defined by

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 + f, \\ \dot{x}_2 &= -3x_1 - 2x_2, \\ y &= x_1. \end{aligned}$$

When the initial state transformation $x = S\bar{x}$ is defined by the identity matrix $S = I = I_2$, the reduced first order system has transfer function $1/(s-1)$ – very little in common with the original system.

4.1.2 An alternative interpretation of projection MOR

For efficient numerical calculations, as well as for the sake of formal mathematical manipulations, a different (though equivalent) interpretation of the projection approach can be introduced.

Let n be the order of the original model, and let $r < n$ be the desired reduced system order. A particular projection can be defined by specifying an n-by-r matrix V and an r-by-n matrix U such that

$$UV = I_r.$$

The reduced system is then defined by substituting $x(t) = V\hat{x}(t)$ into the original equations, and by post-multiplying the resulting differential equation by U on the left, which yields

$$\hat{A} = UAV, \quad \hat{B} = UB, \quad \hat{C} = CV, \quad \hat{D} = D.$$

There are two important messages about this representation of a projection MOR framework. First, the outcome of the procedure cannot be changed much by replacing columns of V with their linear combinations, as well as by replacing rows of U by their linear combinations, as follows from the next statement.

Theorem 4.1 *If matrices V_1, V_2, U_1, U_2 are such that*

(a) $U_1V_1 = U_2V_2 = I_r$,

(b) the linear spans of the columns of V_1 and V_2 are identical, and

(c) the linear spans of the rows of U_1 and U_2 are identical,

then the systems

$$\hat{G}_1 := \left(\begin{array}{c|c} U_1AV_1 & U_1B \\ \hline CV_1 & D \end{array}\right), \quad \hat{G}_2 := \left(\begin{array}{c|c} U_2AV_2 & U_2B \\ \hline CV_2 & D \end{array}\right)$$

have identical transfer matrices.

Proof According to (b), there exists an r-by-r matrix S_v such that $V_2 = V_1S_v$. Similarly, according to (c), there exists an r-by-r matrix S_u such that $U_2 = S_uU_1$. Hence, from (a), $S_uS_v = I$, i.e. $S_u = S_v^{-1}$, and the state space model of $\hat{G}_2$ is obtained from that of $\hat{G}_1$ by replacing its state $\hat{x}_1$ with $\hat{x}_2 = S_u\hat{x}_1$. Since invertible linear transformations of the state vector do not change transfer matrices, the proof is complete. ■

In particular, without loss of generality, one can limit attention only to those matrices V for which $V'V = I_r$. Similarly, one can limit attention only to those matrices U for which $UU' = I_r$, though in this case it would be restrictive to assume that $V'V = I_r$ as well.

To relate the two approaches to projection MOR, note that a pair of matrices U, V satisfying the condition $UV = I_r$ can always be complemented to a pair of mutually inverse matrices

$$S_u = \begin{bmatrix} U \\ \Delta_u \end{bmatrix}, \quad S_v = \begin{bmatrix} V & \Delta_v \end{bmatrix}.$$

If

$$S_v^{-1}AS_v = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad S_v^{-1}B = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \quad CS_v = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix},$$

where $\bar{A}_{11}, \bar{B}_1, \bar{C}_1$ have appropriate dimensions, we have

$$\bar{A}_{11} = UAV, \quad \bar{B}_1 = UB, \quad \bar{C}_1 = CV,$$

i.e. the two projections yield same outcome.

Similarly one can relate the original framework to the coordinate-free one by defining

$$V = S\begin{bmatrix} I_r \\ 0 \end{bmatrix}, \quad U = \begin{bmatrix} I_r & 0 \end{bmatrix} S^{-1},$$

where $x = S\bar{x}$ is the original coordinate transformation.

4.1.3 Projections for other model types

For *discrete time LTI SS models*, projection MOR is defined in exactly the same way as in the continuous time case:

$$x[k+1] = Ax[k] + Bf[k], \quad y[k] = Cx[k] + Df[k]$$

is reduced to

$$\hat{x}[k+1] = UAV\hat{x}[k] + UBf[k], \quad y[k] = CV\hat{x}[k] + Df[k],$$

where $UV = I$.

For *descriptor system models*, the modification is straightforward:

$$E\dot{x}(t) = Ax(t) + Bf(t), \quad y(t) = Cx(t) + Df(t)$$

is replaced with

$$UEV\frac{d}{dt}\hat{x}(t) = UAV\hat{x}(t) + UBf(t), \quad y(t) = CV\hat{x}(t) + Df(t),$$

where the constraint $UV = I$ is dropped, but instead a requirement that $\det(sUEV - UAV)$ is not identically zero is imposed, provided that $\det(sE - A)$ is not identically zero.

For *nonlinear time-invariant state space models*

$$\dot{x}(t) = a(x(t), f(t)), \quad y(t) = h(x(t), f(t)),$$

the assumed transformation from $\hat{x}(t)$ to $x(t)$ and back should be allowed to become nonlinear, i.e.

$$x(t) = V(\hat{x}(t)), \quad \hat{x}(t) = U(x(t)),$$

where $V: \mathbf{R}^r \mapsto \mathbf{R}^n$ and $U: \mathbf{R}^n \mapsto \mathbf{R}^r$ are given differentiable functions satisfying the identity

$$U(V(z)) = z \quad \forall\, z \in \mathbf{R}^r.$$

The resulting model is

$$\frac{d}{dt}\hat{x}(t) = \dot{U}(V(\hat{x}(t)))a(V(\hat{x}(t)), f(t)), \quad y(t) = h(V(\hat{x}(t)), f(t)).$$

While the number of state gets reduced this way, it is not obvious (and is not proven to any degree of generality) that the number of arithmetic operations needed to evaluate $d\hat{x}(t)/dt$ is smaller than the number of operations needed to evaluate $dx(t)/dt$.

For *linear time-varying models*

$$\dot{x}(t) = A(t)x(t) + B(t)f(t), \quad y(t) = C(t)x(t) + D(t)f(t),$$

the projected reduced model is frequently defined as

$$\frac{d}{dt}\hat{x}(t) = U(t)(A(t)V(t) - \dot{V}(t))\hat{x}(t) + U(t)B(t)f(t), \; y(t) = C(t)V(t)\hat{x}(t) + D(t)f(t),$$

where U, V are differentiable functions of time such that $U(t)V(t) = I$ at all moments. Once again, while reduction in the dimension of the state space is indisputable, the situation with complexity of reduced model simulation is not as clear, since the new coefficients $\hat{A}(t), \hat{B}(t), \hat{C}(t)$ may be less regular functions of time, which usually requires extra effort in simulation.

For *general causal input/output models*, where an explicit state description may be unavailable, the *past input/output history*

$$x(t) = (f(\tau)|_{\tau \le t}, y(\tau)|_{\tau \le t})$$

can serve as a replacement. Then the reduced state can be defined by a transformation

$$\hat{x}(t) = U(x(t)) = U(f(\tau)|_{\tau \le t}, y(\tau)|_{\tau \le t}).$$

4.2 System properties preserved under projections

Consider an n-th order state space LTI model

$$\frac{d}{dt}x(t) = Ax(t) + Bf(t), \quad y(t) = Cx(t) + Df(t), \tag{4.1}$$

with transfer matrix

$$G(s) = D + C(sI_n - A)^{-1}B,$$

and its projection reduced r-th model

$$\frac{d}{dt}\hat{x}(t) = UAV\hat{x}(t) + UBf(t), \quad \hat{y}(t) = CV\hat{x}(t) + Df(t), \tag{4.2}$$

with transfer matrix

$$\hat{G}(s) = D + CV(sI_r - UAV)^{-1}UB,$$

where U, V are matrices of dimensions r-by-n and n-by-r respectively, such that $UV = I_r$, and $r < n$. As it was mentioned before, no system property of importance (such as stability, passivity, transfer matrix values, etc.) is guaranteed to be preserved by a generic projection of this type. However, when the projection matrices U, V are properly defined, some features of the original system can be transfered to the reduced one.

4.2.1 Preservation of transfer matrix moments

This subsection provides *sufficient* conditions for preservation of *transfer matrix moments* under projection model reduction. Here by the *moments* of a transfer matrix $H = H(s)$ at a given point s_0, which is *not* a pole of $H(\cdot)$, we mean the values $H^{(k)}(s_0)$ of its derivatives and its own value $H(s_0) = H^{(0)}(s_0)$ at $s = s_0$.

Theorem 4.2 *Consider system (4.1) and its projection reduced model (4.2). Let $s_0 \in \mathbf{C}$ be a complex number which is* not *an eigenvalue of A or UAV.*

(a) If, for some column vector $\bar{f}$, $(s_0 I_n - A)^{-k-1}B\bar{f}$ belongs to the set $\mathcal{R}(V)$ of all linear combinations of columns of V for all $k = 0, \ldots, N$ then

$$G^{(k)}(s_0)\bar{f} = \hat{G}^{(k)}(s_0)\bar{f} \quad \text{for} \quad k = 0, \ldots, N.$$

(b) If $\bar{q}C(s_0 I_n - A)^{-k-1}$ is a linear combination of the rows of U for some row vector $\bar{q}$ for all $k = 0, \ldots, N$ then

$$\bar{q}G^{(k)}(s_0) = \bar{q}\hat{G}^{(k)}(s_0) \quad \text{for} \quad k = 0, \ldots, N.$$

This observation is frequently used in determining the projection matrices U, V: the rows of U and the columns of V are chosen as bases of the linear subspaces spanned by the real and imaginary parts of the rows of

$$C(j\omega_i^c I_n - A)^{-k-1}, \quad k = 0, \dots, N_i^c,$$

and the columns of

$$(j\omega_i^b I_n - A)^{-k-1}B, \quad k = 0, \dots, N_i^b,$$

respectively, where ω_i^c and ω_i^b are selected "important" frequencies, and the maximal powers N_i^c, N_i^b indicate the degree of accuracy desired at ω_i^c, ω_i^b. The resulting projection algorithms provide a numerically robust implementation of the *moments matching* approach to model reduction, to be discussed in future lectures.

Proof Without loss of generality, assume that $D = 0$. To prove (a), note that

$$G^{(k)}(s_0) = k!C(s_0 I_n - A)^{-k-1}B \quad (k = 0, 1, \dots),$$

and

$$\hat{G}^{(k)}(s_0) = k!CV(s_0 I_r - UAV)^{-k-1}UB \quad (k = 0, 1, \dots).$$

Note that the vectors

$$x_k = (s_0 I_n - A)^{-k-1}B\bar{f}, \quad (k = 0, 1, \dots)$$

are uniquely defined by the recursive linear equations

$$s_0 x_0 = Ax_0 + B\bar{f}, \quad s_0 x_k = Ax_k + x_{k-1}, \quad (k = 1, \dots).$$

By the assumption, $x_k = V\hat{x}_k$ for $k = 0, \dots, N$ for some vectors $\hat{x}_k$. Hence

$$s_0 V\hat{x}_0 = AV\hat{x}_0 + B\bar{f}, \quad s_0 V\hat{x}_k = AV\hat{x}_k + V\hat{x}_{k-1}, \quad (k = 1, \dots, N).$$

Multiplying these equalities by U on the left yields

$$s_0\hat{x}_0 = \hat{A}\hat{x}_0 + \hat{B}\bar{f}, \quad s_0\hat{x}_k = \hat{A}\hat{x}_k + \hat{x}_{k-1}, \quad (k = 1, \dots, N),$$

where $\hat{A} = UAV$ and $\hat{B} = UB$. Hence

$$\hat{x}_k = (s_0 I_r - \hat{A})^{-k-1}\hat{B}\bar{f}, \quad (k = 0, \dots, N),$$

which in turn yields

$$C(s_0 I_n - A)^{-k-1}B\bar{f} = Cx_k = CU\hat{x}_k = \hat{C}\hat{x}_k = \hat{C}(s_0 I_r - \hat{A})^{-k-1}\hat{B}\bar{f}$$

for $k = 0, \ldots, N$, thus proving (a).

The proof of (b) is similar, and uses

$$p_k = \bar{q}C(s_0 I_n - A)^{-k-1}$$

in place of x_k. By the assumption, $p_k = \hat{p}_k U$ for $k = 0, \ldots, N$ for some row vectors $\hat{p}_k$. Hence

$$s_0 \hat{p}_0 U = \hat{p}_0 U A + \bar{q}C, \quad s_0 \hat{p}_k U = \hat{p}_k U A + \hat{p}_{k-1} U, \quad (k = 1, \ldots, N).$$

Multiplying these equalities by V on the right yields

$$s_0 \hat{p}_0 = \hat{p}_0 \hat{A} + \bar{q}\hat{C}, \quad s_0 \hat{p}_k = \hat{p}_k \hat{A} + \hat{p}_{k-1}, \quad (k = 1, \ldots, N),$$

where $\hat{C} = CV$. Hence

$$\hat{p}_k = \bar{q}C(s_0 I_r - \hat{A})^{-1}, \quad (k = 0, \ldots, N),$$

which yields

$$\bar{q}C(s_0 I_n - A)^{-k-1} B = p_k B = \hat{p}_k U B = \hat{p}_k \hat{B} = \bar{q}\hat{C}(s_0 I_r - \hat{A})^{-k-1} \hat{B}$$

for $k = 0, \ldots, N$, thus proving (b). ■

The results of this subsection apply equally to discrete time LTI state space models. They can also be extended to the case of descriptor models, to yield the following statement, which is proven by a similar argument.

Theorem 4.3 *Consider the descriptor system model*

$$E\frac{d}{dt}x(t) = Ax(t) + Bf(t), \quad y(t) = Cx(t) + Df(t),$$

and its projection reduced model

$$\hat{E}\frac{d}{dt}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}f(t), \quad \hat{y}(t) = \hat{C}\hat{x}(t) + Df(t),$$

where

$$\hat{A} = UAV, \quad \hat{E} = UEV, \quad \hat{B} = UB, \quad \hat{C} = CV,$$

and U, V are matrices of dimensions r-by-n and n-by-r respectively, such that $r < n$. Let $s_0 \in \mathbf{C}$ be a complex number such that both matrices $s_0 E - A$ and $s_0 \hat{E} - \hat{A}$ are invertible .

(a) If, for some column vector $\bar{f}$, $[(s_0E-A)^{-1}E]^k(s_0E-A)^{-1}B\bar{f}$ belongs to the set $\mathcal{R}(V)$ of all linear combinations of columns of V for all $k=0,\dots,N$ then

$$G^{(k)}(s_0)\bar{f}=\hat{G}^{(k)}(s_0)\bar{f} \quad \text{for} \quad k=0,\dots,N,$$

where

$$G(s)=C(sE-A)^{-1}B+D, \quad \hat{G}(s)=\hat{C}(s\hat{E}-\hat{A})^{-1}\hat{B}+\hat{D}.$$

(b) If $\bar{q}C(s_0E-A)^{-1}[E(s_0E-A)^{-1}]^k$ is a linear combination of the rows of U for some row vector $\bar{q}$ for all $k=0,\dots,N$ then

$$\bar{q}G^{(k)}(s_0)=\bar{q}\hat{G}^{(k)}(s_0) \quad \text{for} \quad k=0,\dots,N.$$

4.2.2 Stability preservation

In most applications, a reduced model of a stable system is required to be stable. However, stability preservation does not come automatically in projection model reduction.

The following result provides a sufficient condition for stability of projection reduced systems in the continuous time case. Remember that a controllable and observable state space model (4.1) defines a stable causal system if and only if A is a Hurwitz matrix (all eigenvalues have negative real part), or, equivalently, if there exists a symmetric strictly positive definite matrix $P=P'>0$ such that the *Lyapunov equality*

$$PA+A'P=-R \tag{4.3}$$

is satisfied for some positive semidefinite symmetric matrix $R\geq 0$ such that the pair (A,R) is controllable (note that the controllability is implied, in particular, when R is strictly positive definite). Such $P=P'$ defines an energy-like quantity $V(x(t))=x(t)'Px(t)$ which is guaranteed not to increase when the external signal $f=f(t)$ equals zero. In many applications, $P=P'$ is readily available from the physical laws of energy conservation and dissipation.

Theorem 4.4 *Consider system (4.1) and its projection reduced model (4.2). Let $P=P'$ and V be such that*

$$V'PV>0, \quad V'(PA+A'P)V<0.$$

If U is defined by

$$U=(V'PV)^{-1}V'P$$

then $\hat{A}=UAV$ is a Hurwitz matrix satisfying the Lyapunov inequality

$$\hat{P}\hat{A}+\hat{A}'\hat{P}<0 \quad \text{for} \quad \hat{P}=V'PV.$$

Proof The proof is based on a straightforward verification of the identity

$$\hat{P}\hat{A} = V'PAV.$$

■

The theorem suggests that stability of the reduced system can be guaranteed easily by limiting the freedom of selecting U for a given V: to use the result, one has U uniquely defined by V and a conserved energy measure matrix P. Note that condition $V'(PA + A'P)V < 0$ can be replaced by the weaker one:

$$V'(PA + A'P)V = -\hat{R},$$

where $\hat{R} = \hat{R}' \geq 0$ and the pair $(\hat{A}, \hat{R})$ is controllable.

For descriptor models, similar results are available. Remember that a certificate of (marginal) stability for a descriptor system

$$E\frac{d}{dt}x(t) = Ax(t) + Bf(t), \quad y(t) = Cx(t) + Df(t) \tag{4.4}$$

is given by a symmetric matrix $P = P'$ such that $E'PE \geq E'E$ and

$$E'PA + A'PE \leq 0. \tag{4.5}$$

Theorem 4.5 *Let $P = P'$ be a symmetric n-by-n matrix and V be an n-by-r matrix such that*

$$\hat{P} = V'E'PEV > 0 \quad \text{and} \quad V'(E'AP + PA'E')V < 0.$$

Let

$$U = (V'E'PEV)^{-1}V'E'P.$$

Then the reduced system

$$\hat{E}\frac{d}{dt}\hat{x}(t) = \hat{A}x(t) + \hat{B}f(t), \quad \hat{y}(t) = \hat{C}\hat{x}(t) + Df(t),$$

where

$$\hat{E} = UEV = I_r, \quad \hat{A} = UAV, \quad \hat{B} = UB, \quad \hat{C} = CV,$$

is stable, and has a Lyapunov function

$$\hat{V}(\hat{x}(t)) = \hat{x}(t)'\hat{P}\hat{x}(t)$$

which decreases monotonically for system solutions with $f(t) = 0$.

Proof The proof of this theorem is a straightforward verification of the identity

$$\hat{P}\hat{A} = V'E'PAV.$$

■

Similar stability preservation results are also available for discrete time models

$$x[k+1] = Ax[k] + Bf[k], \quad y[k] = Cx[k] + Df[k], \tag{4.6}$$

where the stability of a minimal state space model (4.6) is equivalent to A being a Schur matrix (all eigenvalues strictly inside the unit disc), and a stability certificate is usually given in the form of a symmetric positive definite matrix $P = P' > 0$ such that the *discrete time Lyapunov inequality*

$$P - A'PA > 0$$

is satisfied.

Theorem 4.6 *Consider a discrete time LTI model (4.6). Let $P = P' \geq 0$ and V be matrices of dimensions n-by-n and n-by-r respectively, such that*

$$V'PV > 0, \quad V'(A'PA - P)V < 0.$$

If U is defined by

$$U = (V'PV)^{-1}V'P$$

then $\hat{A} = UAV$ is a Schur matrix satisfying the Lyapunov inequality

$$\hat{A}'\hat{P}\hat{A} - \hat{P} < 0 \quad \text{for} \quad \hat{P} = V'PV,$$

and hence the projection reduced model

$$\hat{x}[k+1] = UAV\hat{x}[k] + UBf[k], \quad \hat{y}[k] = CV\hat{x}[k] + Df[k]$$

is stable.

Proof The proof is based on a straightforward verification of the inequality

$$PV(V'PV)^{-1}V'P \leq P.$$

In fact, the assumption $V'(A'PA - P)V < 0$ in the theorem formulation can be relaxed to the weaker one

$$V'PV - V'A'PV(V'PV)^{-1}V'PAV > 0.$$

■

4.2.3 L2 gain and passivity preservation

Let us call a stable LTI system model with a proper rational transfer matrix $H = H(s)$ *contracting* if

$$\sigma_{\max}(H(j\omega)) = \|H(j\omega)\| < 1 \ \ \forall\, \omega \in \mathbf{R}.$$

A formally different, but actually very similar definition concerns *passivity*. A stable LTI system model with a proper *square* rational transfer matrix $H = H(s)$ is called *passive* if

$$H(j\omega) + H(j\omega)' < 0 \ \ \forall\, \omega \in \mathbf{R},$$

where the prime sign denotes Hermitian conjugation.

Both contractivity and passivity serve as frequency domain conditions for *input/output energy conservation*, having an interpretation that one gets less energy *out* of a system than the amount that was put *into* the system. In circuits applications, for example, contractivity applies to voltage in/voltage out subsystems with large input resistance and small output resistance, so that the supplied/extracted energies equal the integrals of input/output squared, while passivity applies to single-port blocks with voltage and current serving as input and output, so that the difference between supplied and extracted energies equals the integral of the input/output product. Contractivity and passivity are important because feedback interconnections of energy preserving systems preserve the total energy, which implies stability.

As it can be expected, all cases of passivity and contractivity of state space models come with a positive definite quadratic Lyapunov function representing a conserved (dissipated) energy quantity. Mathematically, such statements are special cases of the *Kalman-Yakubovich-Popov Lemma*, which will be presented here without a proof and with a simplifying assumption of strict properness of $H(s)$.

Theorem 4.7 *Let*

$$\dot{x}(t) = Ax(t) + Bf(t), \quad y(t) = Cx(t)$$

be a minimal model of an n-th order stable strictly proper causal system with transfer matrix $H = H(s)$. Then

(a) $H = H(s)$ is contracting if and only if there exists a symmetric positive definite matrix $P = P' > 0$ such that the Riccati inequality

$$PA + A'P + C'C + PBB'P < 0$$

is satisfied;

(b) $H = H(s)$ *is passive if and only if there exists a symmetric positive definite matrix* $P = P' > 0$ *such that*

$$C = B'P \quad \text{and} \quad PA + A'P < 0.$$

The matrix inequality from (a) means that

$$\frac{dV(x(t))}{dt} \leq |f(t)|^2 - |y(t)|^2$$

for all solutions of the system equations, where

$$V(\bar{x}) = \bar{x}' P \bar{x} \geq 0$$

is the "potential energy" of the system. Similarly, the matrix inequality from (b) means that

$$\frac{dV(x(t))}{dt} \leq 2f(t)'y(t).$$

It turns out that the same way of selecting U when P and V are given, that was used to preserve stability in projection model reduction, can be employed to preserve contractivity and passivity as well.

Theorem 4.8

(a) *If* $P = P'$ *is such that*

$$\hat{P} = V'PV > 0 \;\; \text{and} \;\; V'(PA + A'P + C'C + PBB'P)V < 0,$$

then the projection reduced model

$$\frac{d}{dt}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}f(t), \quad \hat{y}(t) = \hat{C}\hat{x}(t), \tag{4.7}$$

where

$$\hat{A} = UAV, \quad \hat{B} = UB, \quad \hat{C} = CV, \quad U = (V'PV)^{-1}V'P,$$

is contracting, and its coefficients satisfy the Riccati inequality

$$\hat{P}\hat{A} + \hat{A}'\hat{P} + \hat{C}'\hat{C} + \hat{P}\hat{B}\hat{B}'\hat{P} < 0.$$

(b) *If* $P = P'$ *is such that*

$$CV = B'PV, \quad \hat{P} = V'PV > 0 \;\; \text{and} \;\; V'(PA + A'P)V < 0$$

then the projection reduced model (4.7) is passive, and its coefficients satisfy

$$\hat{C} = \hat{B}'\hat{P} \quad \text{and} \quad \hat{P}\hat{A} + \hat{A}'\hat{P} < 0.$$