Massachusetts Institute of Technology

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6.242, Fall 2004: MODEL REDUCTION *

System approximation via orthonormal decomposition[1]

This lecture discusses optimal approximation of transfer functions by lower order rational transfer functions with a given set of poles. While the technique does not really qualify as "model reduction", it can be quite useful as a preliminary step in a model reduction algorithm for large scale LTI models.

8.1 Linear expansions of functions

In this section we recall the basic definitions associated with orthogonal decompositions.

8.1.1 Linear decompositions in function spaces

A common way of representing high-dimensional (or infinite dimensional) vector data, such as signals, images, or other functions, is linear decomposition into elements of a given set of "simple", or *basis* functions. For example, Fourier transforms of smooth functions $f = f(t)$ of time $t \in \mathbf{R}$ which are decaying fast enough as $t \to \pm\infty$ can be viewed as a way of representing f as integrals of harmonic oscillations $f_\omega(t) = \cos(\omega t)$ and $g_\omega(t) = \sin(\omega t)$, according to

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\tilde{f}(j\omega)d\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\cos(\omega t)f_r(\omega)d\omega - \frac{1}{\pi}\int_0^{\infty}\sin(\omega t)f_i(\omega)d\omega, \end{aligned}$$

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where $f_r(\omega)$ and $f_i(\omega)$ are the real and imaginary parts of the Fourier transform

$$\tilde{f}(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t) dt$$

of $f = f(t)$.

Another such example is representation of almost periodic continuous signals $f = f(t)$, i.e. functions $f : \mathbf{R} \mapsto \mathbf{R}$ such that

$$\sup_{t \in \mathbf{R}} |f(t)| < \infty,$$

and

$$\inf_{T \geq 1} \sup_{t \in \mathbf{R}} |f(t) - f(t - T)| = 0,$$

as sums

$$f(t) = \sum_{k=1}^{\infty} f_k e^{j\omega_k t}, \quad \text{where } \sum_{k=1}^{\infty} |f_k| < \infty,$$

of a *countable* number of harmonic oscillations.

8.1.2 Orthonormal families of functions

While the bases of harmonic oscillations are working well in many applications, other bases produce poor results. For example, linear combinations of functions $g_k(t) = t^{2k}$ (which includes all polynomials of t^2) can be used to approximate uniformly arbitrary continuous functions $g = g(t)$ of time $t \in [0, 1]$. However, trying to represent a "nice" function $f(t) = (1 + at^2)^2$, where $a \geq 0$ is a parameter, as a linear expansion

$$f(t) = \sum_{k=0}^{\infty} f_k g_k(t) = \sum_{k=0}^{\infty} f_k t^{2k}, \quad t \in [0, 1],$$

yields $f_k = (-a)^k$, which is not good when $a \gg 1$, as the resulting representation of $f(t)$ involves relative accuracy and conditioning loss associated with obtaining a small difference in substracting one large number from another.

To avoid such possibility, it is usually a good idea to use the so-called *orthonormal* bases. In general, orthonormality is defined with respect to a *scalar product*, which is a function $\sigma : \mathcal{V} \times \mathcal{V} \mapsto \mathbf{C}$ defined on the set $\mathcal{V} \times \mathcal{V}$ of pairs (v_1, v_2) of functions from a certain class (closed with respect to the operations of addition and scaling), satisfying the following conditions:

(a) $\sigma(v_1, v_2) = \sigma(v_2, v_1)$ for all $v_1, v_2 \in \mathcal{V}$ (symmetry);

(b) $\sigma(v, c_1v_1 + c_2v_2) = c_1\sigma(v, v_1) + c_2\sigma(v, v_2)$ for all $v, v_1, v_2 \in \mathcal{V}$, $c_1, c_2 \in \mathbf{R}$ (linearity);

(c) $\sigma(v, v) > 0$ for all $v \in \mathcal{V}$, $v \not 0$ (positivity).

A family of functions $g_i \in \mathcal{V}$ is called *orthonormal* with respect to σ if

$$\sigma(g_i, g_k) = \delta_{ik} = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

For example, in the case of working with functions of a scalar variable $\theta \in \mathbf{R}$ (where, in system applications, θ could be "time" t or "frequency" ω), the scalar product is typically defined by

$$\sigma(v_1, v_2) = \int_{\mathbf{R}} v_1(\theta)' v_2(\theta)\rho(\theta)dt,$$

where $\rho: \mathbf{R} \mapsto [0, \infty)$ is a given *weight* function. The resulting orthonormality condition becomes

$$\int_{\mathbf{R}} g_i(\theta)' g_k(t)dt = \delta_{ik}.$$

However, scalar products are not always defined as integrals. For example, when $\mathcal{V}$ is the class of all almost periodic functions (defined earlier in this lecture), the typical scalar product is defined by

$$\sigma(v_1, v_2) = \lim_{T\to\infty} \frac{1}{T}\int_0^T v_1(t)' v_2(t)dt.$$

It is interesting to note that the set of all exponents $g_\omega(t) = e^{j\omega t}$ is an uncountable orthonormal family with respect to σ.

8.1.3 Orthonormal decompositions

The following classical theorem states the benefits of orthonormality, which includes certain optimality of linear expansions, a-priori estimates for the amplitudes of expansion coefficients, and explicit formulae for their calculation.

Theorem 8.1 *Let $\{g_k\}_{k=0}^N$ be a family of functions in $\mathcal{V}$ which is orthonormal with respect to a scalar product $\sigma: \mathcal{V} \times \mathcal{V} \mapsto \mathbf{R}$. For every $v \in \mathcal{V}$, define*

$$\hat{v} = \sum_{k=0}^N v_k g_k, \quad \text{where} \quad v_k = \sigma(g_k, v).$$

Then

(a) $g=\hat{v}$ is the argument of minimum of $\sigma(g-v,g-v)$ over the set of all linear combinations g of g_k;

(b) the sum of squares of v_k does not exceed $\sigma(v,v)$.

The proof of the theorem is omitted as being standard. The function $\hat{v}$ is considered as the optimal (with respect to s) approximation of v by linear combinations of functions g_k. Note how the the choice of the scalar product σ is related to performance of optimal approximation: the quality guarantee is given in terms of the norm $\|v\|_\sigma=\sigma(v,v)^{1/2}$ defined by σ. In particular, it is important to use scalar products which are adapted to a particular application.

For example, for across-the-spectrum approximation of strictly proper transfer functions, the standard scalar product

$$\sigma(G_1,G_2)=\frac{1}{\pi}\int_0^\infty G_1(j\omega)'G_2(j\omega)d\omega$$

may be appropriate. In contrast, for approximation of proper transfer functions which should be best in the low frequency region $|\omega|<\omega_0$, the scalar product

$$\sigma(G_1,G_2)=\frac{1}{\pi}\int_0^\infty G_1(j\omega)'G_2(j\omega)\frac{d\omega}{\omega^2+\omega_0^2}$$

is likely to work better.

8.1.4 Orthogonalization

In many applications it is natural to start with a sequence of functions $w_0,w_1,w_2,\dots$ which are *not* orthonormal with respect to the desired scalar product σ. For example, when approximating functions $v:\ [0,1]\mapsto\mathbf{R}$ by low degree polynomials, it is reasonable to use

$$\sigma(v_1,v_2)=\int_0^1 v_1(t)v_2(t)dt,\quad w_k(t)=t^k.$$

To make finding of best least squares approximations efficient and robust, it is desirable to *orthogonalize* the original sequence $w_0,w_1,w_2,\dots,$ by finding an orthonormal family

$g_0, g_1, g_2, \ldots$ such that

$$\begin{aligned} g_0 &= c_{00}w_0, \\ g_1 &= c_{10}w_0 + c_{11}w_1, \\ &\vdots \\ g_k &= \sum_{i=0}^{k} c_{ki}w_i, \\ &\vdots \end{aligned}$$

This can be achieved by using the Gram-Schmidt orthogonalization procedure

$$\begin{aligned} g_0 &= \sigma(w_0, w_0)^{-1/2}w_0, \\ \Delta_k &= w_{k+1} - \sum_{i=0}^{k} \sigma(g_i, w_{k+1})g_i, \\ g_{k+1} &= \sigma(\Delta_k, \Delta_k)^{-1/2}\Delta_k, \end{aligned}$$

The procedure (which is implemented in MATLAB as the *Cholesky decomposition* `chol.m`) is well defined when functions $w_0, \ldots, w_n$ are linearly independent for all n, i.e. when the only identically zero linear combination of w_i is the one with zero coefficients.

For example, Gram-Schmidt orthogonalization of $w_k(t) = t^k$ with respect to the scalar product

$$\sigma(v_1, v_2) = \int_0^1 v_1(t)v_2(t)dt$$

yields

$$g_0(t) \equiv 1, \quad g_1(t) = \sqrt{3}(2t - 1), \quad \ldots$$

8.2 Orthonormal expansions of stable transfer functions

The objective of this section is to introduce convenient orthonormal bases of elementary transfer functions for accurate and robust approximation of complex LTI systems.

8.2.1 Scalar products for transfer functions

The *standard* scalar product for the class of strictly proper real rational transfer functions without poles on the imaginary axis is defined by

$$\sigma(G_1, G_2) = \sigma_1(G_1, G_2) = \frac{1}{\pi}\int_0^\infty G_1(j\omega)'G_2(j\omega)d\omega = \int_{-\infty}^{\infty} g_1(t)'g_2(t)dt,$$

where g_i are the stable (but not necessarily causal) impulse responses of G_i. For example,

$$\sigma_1\left(\frac{1}{s+1},\frac{1}{s+a}\right)=\begin{cases}1/(1+a), & a>0,\\ 0, & a<0,\\ ?, & a=0,\end{cases}$$

where "?" means "not defined".

Alternative scalar products are typically defined by

$$\sigma(G_1,G_2)=\sigma_\psi(G_1,G_2)=\frac{1}{\pi}\int_0^\infty G_1(j\omega)'G_2(j\omega)|\psi(j\omega)|^2 d\omega,$$

where $\psi\not\equiv 0$ is a given proper rational transfer function without poles on the imaginary axis.

For such scalar products, verification of orthogonality can be facilitated by the following observation.

Lemma 8.1 *If H is a stable transfer function of relative degree larger than 1 then*

$$\frac{1}{\pi}\int_0^\infty H(j\omega)d\omega=0.$$

For example, using the lemma with

$$H(s)=\frac{1}{(s+1)(-s-2)}$$

yields

$$\sigma_1\left(\frac{1}{s+1},\frac{1}{s-2}\right)=\frac{1}{\pi}\int_0^\infty\frac{1}{(j\omega+1)(-j\omega-2)}d\omega=0.$$

8.2.2 Canonical orthonormal families of stable transfer functions

In this subsection we construct an important parameterized orthonormal family in the set of strictly proper stable transfer functions. This family will be refered to as the *canonical* one.

Theorem 8.2 *Let $a_k>0$ for $k=0,1,2,\dots$ be a sequence of positive numbers. Then the sequence of transfer functions*

$$\theta_0=\frac{\sqrt{2a_0}}{s+a_0},\quad \theta_k=\frac{\sqrt{2a_k}}{s+a_k}\prod_{i=0}^{k-1}\frac{s-a_i}{s+a_i}$$

is orthonormal with respect to the standard scalar product.

Proof Use the lemma to establish that $\sigma_1(\theta_i, \theta_k) = 0$ for $i \neq k$. To check normalization, note first that

$$\left|\frac{s-a_i}{s+a_i}\right| = 1 \quad \text{for} \quad s = j\omega.$$

Hence

$$\sigma_1(\theta_k, \theta_k) = \frac{1}{\pi}\int_0^\infty \frac{2a_k d\omega}{|j\omega + a_k|^2} = 2a_k \int_0^\infty e^{-2a_k t} dt = 1.$$

■

When all numbers a_k are different, the sequence θ_k from the theorem is the result of Gram-Schmidt orthogonalization of the sequence

$$w_k(s) = \frac{1}{s+a_k}.$$

In general, θ_k can be interpreted as the result of orthogonalizing the sequence

$$w_k(s) = \prod_{i=0}^{k} \frac{1}{s+a_i}.$$

It is important to know when functions θ_k actually form a *basis* in the set of all stable transfer functions with square integrable impulse responses. The following result, stated without a proof here, answers this question.

Theorem 8.3 *The family $\{\theta_k\}_{k=0}^\infty$ is a basis in the set of stable transfer functions if and only if*

$$\sum_{k=0}^\infty a_k = \infty.$$

8.2.3 Explicit formulae for canonical decompositions

Let us refer to the optimal linear decomposition

$$G(s) = \sum_{k=0}^{N} g_k \theta_k(s),$$

where $\{\theta_k\}_{k=0}^\infty$ is a canonical basis defined by a non-summable sequence of positive numbers a_k, as the *canonical* one.

Though the coefficients $g_k = \sigma_1(\theta_k, G)$ can be defined as the corresponding integrals, actually computing them according to the integration formula would be extremely inefficient when a state space model of G is available.

The following theorem provides a more efficient approach.

Theorem 8.4 *Let* $\{a_k\}_{k=0}^\infty$ *be a non-summable sequence of positive numbers. let* $\{\theta_k\}_{k=0}^\infty$ *be the corresponding canonical basis in the space of all stable strictly proper transfer functions. Then the coefficients* g_k *of the canonical decomposition*

$$G(s) \approx \sum_{k=0}^{N} g_k \theta_k(s),$$

are defined by the recursion

$$\begin{aligned} G_0(s) &= G(s), \\ g_k &= \sqrt{2a_k} G_k(a_k), \\ G_{k+1}(s) &= \frac{s+a_k}{s-a_k}\left(G_k(s) - g_k \frac{2a_k}{s+a_k}\right). \end{aligned}$$

Moreover, if $G(s) = C(sI - A)^{-1}B$ *then* G_k*,* g_k *can be computed recursively by*

$$\begin{aligned} B_0 &= B, \\ F_k &= (a_k I - A)^{-1} B_k, \\ g_k &= C F_k, \\ B_{k+1} &= -(a_k I + A) F_k. \end{aligned}$$

The proof of the theorem is by a straightforward inspection. Note that the sequence of vectors B_k is defined by the recursion

$$B_{k+1} = \tilde{A}_k B_k, \quad \tilde{A}_k = -(a_k I + A)(a_k I - A)^{-1}.$$

If A is a Hurwitz matrix satisfying Lyapunov inequality

$$PA + A'P \leq 0,$$

where $P = P' > 0$, then $\tilde{A}_k$ satisfies the discrete time Lyapunov inequality

$$P \geq \tilde{A}_k' P \tilde{A}_k.$$

Hence $B_k' P B_k$ is monotonically non-increasing, and vector B_k remains bounded as $k \to \infty$.

8.2.4 Error of cannonical optimal approximations

The quality of canonical optimal approximations

$$G(s) \approx \hat{G}_N(s) = \sum_{k=0}^{N} g_k \theta_k(s)$$

depends significantly on the smoothness of $G(j\omega)$ as a function of ω. The following upper bound for the approximation error is available when $a_k \equiv a > 0$ is a constant sequence.

Theorem 8.5 *Assume that the 2π-periodic function*

$$\phi(\tau) = \frac{a + j\tan(\tau/2)}{a} G(ja\tan(\tau/2))$$

is q times differentiable, and its q-th derivative $\phi^{(q)}$ satisfies

$$M_q = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi^{(q)}(\tau)|^2 d\tau < \infty.$$

Then

$$\|G - \hat{G}_N\|_\infty \leq (2q-1)^{-0.5} M_q^{0.5} N^{0.5-q}.$$

Proof The standard linear fractional substitution

$$z = \frac{s+a}{s-a} = e^{-j\tau}, \quad s = a\frac{z+1}{z-1} = j\omega = j\tan(\tau/2),$$

transforms the orthonormal decomposition

$$G(s) = G(j\omega) = \frac{\sqrt{2a}}{s+a}\sum_{k=0}^{\infty} g_k \left(\frac{s-a}{s+a}\right)^k$$

into

$$G(j\tan(\tau/2)) = \frac{a}{a + j\tan(\tau/2)}\phi(\tau) = \frac{a}{a + j\tan(\tau/2)}\sum_{k=0}^{\infty} \phi_k e^{jk\tau},$$

where

$$\phi_k = \sqrt{\frac{2}{a}} g_k.$$

By construction,

$$\|G - \hat{G}_N\|_\infty \leq \epsilon_N = \max_\tau |\phi(\tau) - \hat{\phi}_N(\tau)|,$$

where

$$\hat{\phi}_N(\tau) = \sum_{k=0}^{N} \phi_k e^{jk\tau}.$$

To give an upper bound for ϵ_N, note that

$$\phi^{(q)}(\tau) = \sum_{k=0}^{\infty} (jk)^q e^{jk\tau},$$

and hence

$$(jk)^q \phi_k = \int_{-\pi}^{\pi} e^{-jk\tau} \phi^{(q)}(\tau) d\tau,$$

which, by the othonormality of the complex exponents $e^{jk\tau}$, implies

$$\sum_{k=0}^{\infty} k^{2q} |\phi_k|^2 \leq M_q.$$

Combining this with the fact that

$$\sum_{k=N+1}^{\infty} k^{-2q} \leq \int_N^{\infty} x^{-2q} dx = \frac{1}{(2q-1)N^{2q-1}},$$

we get

$$\begin{aligned}
\epsilon_N &\leq \sum_{k=N+1}^{\infty} |\phi_k| \\
&= \sum_{k=N+1}^{\infty} (k^q \phi_k)(k^{-q}) \\
&\leq \left(\sum_{k=N+1}^{\infty} k^{2q} \phi_k^2 \right)^{1/2} \left(\sum_{k=N+1}^{\infty} k^{-2q} \right)^{1/2} \\
&\leq M_q^{0.5} (2q-1)^{-0.5} N^{0.5-q}.
\end{aligned}$$

■

Theorem 8.5 is frequently used to get a rough understanding of asymptic behavior of singular numbers of a given distributed LTI system $G = G(s)$. Since the k-th singular number is a *lower* bound of H-Infinity error of approximating G by a system of order less than k, the k-th singular will decrease at least as fast as $ck^{0.5-q}$, where q is the number square integrable derivatives of the corresponding ϕ function.