Massachusetts Institute of Technology

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6.242, Fall 2004: MODEL REDUCTION *

Model reduction via convex optimization[1]

This lecture discusses methods for deriving reduced LTI models by solving a convex optimization problem.

9.1 Convex optimization: an introduction

This section presents basic definitions and elementary results of convex optimization.

9.1.1 A hierarchy of optimization setups

Roughly speaking, an "optimization problem" is the task of finding an element $x = x_o$ of a given set X for which the value $\Phi(x)$ of a given function $\Phi: \ X \mapsto \mathbf{R}$ is minimal. Alternatively, the objective could be to find an element $x_\gamma \in X$ such that $\Phi(x_\gamma h) < \gamma$, where $\gamma \in \mathbf{R}$ is a given threshold, or to give evidence to non-existence of such x_γ does not exist.

Among the most familiar optimization problems is the so-called *linear-quadratic optimization*, which is essentially the task of minimizing a quadratic form

$$\Phi(x) = x'Qx - 2F'x,$$

where $Q = Q' > 0$ is a given real symmetric positive definite n-by-n matrix, and $F \in \mathbf{R}^n$ is a given column n-vector, over $X = \{x\} = \mathbf{R}^n$. The minimum of $-F'Q^{-1}F$ is achieved at $x = x_o = Q^{-1}F$.

[1]Version of October 27, 2004

Another optimization setup exploited earlier in these lectures is the "matrix rank reduction problem", which is the task of finding a matrix $x \in \mathbf{R}^{n\times m}$ of rank strictly less than a given number r such that $\|x-M\|$ is minimal, where $M \in \mathbf{R}^{n\times m}$ is given, and $\|\Delta\|$ denotes the largest singular value (operator norm) of Δ. The solution of the matrix rank reduction problem, though not as simple as that of the linear quadratic optimization, is also quite explicit in terms of standard linear algebra operations.

Convex optimization is another example of an optimization task which, in principle, can be solved efficiently, though the convex optimization algorithms are usually more complicated than those for numerical linear algebra. While a typical model reduction is not given in the form of a convex optimization, there are some ways of modifying the problem to fit within the convex optimization framework.

9.1.2 Convex Sets

A subset Ω of $V = \mathbf{R}^n$ is called *convex* if

$$cv_1 + (1-c)v_2 \in \Omega \quad \text{whenever} \quad v_1, v_2 \in \Omega,\ c \in [0,1].$$

In other words, a set is convex whenever the line segment connecting any two points of Ω lies completely within Ω.

In many applications, the elements of Ω are, formally speaking, not vectors but other mathematical objects, such as matrices, polynomials, etc. What matters, however, is that Ω is a subset of a set V such that a one-to-one correspondence between $\mathbf{R}^n$ and V is established for some n. We will refer to V as a (real finite dimensional) *vector space*, while keeping in mind that V is the same as $\mathbf{R}^n$ for some n. For example, the set $\mathbf{S}^n$ of all symmetric n- by-n matrices is a vector space, because of the natural one-to-one correspondence between $\mathbf{S}^n$ and $\mathbf{R}^{n(n+1)/2}$.

Using this definition directly, in some situations it would be rather difficult to check whether a given set is convex. The following simple statement is of a great help.

Lemma 9.1 *Let K be a set of* affine functionals *on $V = \mathbf{R}^n$, i.e. elements $f \in K$ are functions $f:\ V \to \mathbf{R}$ such that*

$$f(cv_1 + (1-c)v_2) = cv_1 + (1-c)v_2 \quad \forall\ c \in \mathbf{R}, \quad v_1, v_2 \in V.$$

Then the subset Ω of V defined by

$$\Omega = \{v \in V:\ f(v) \geq 0 \ \ \forall\ f \in K\}$$

is convex.

In other word, any set defined by linear inequalities is convex.

Proof Let $v_1, v_2 \in \Omega$ and $c \in [0,1]$. Since $f(v_1) \geq 0$ and $f(v_2) \geq 0$ for all $f \in K$, and $c \geq 0$ and $1 - c \geq 0$, we conclude that

$$f(cv_1 + (1-c)v_2) = cf(v_1) + (1-c)f(v_2) \geq 0$$

for all $f \in K$. Hence $cv_1 + (1-c)v_2 \in K$. ■

Here is an example of how Lemma 9.1 can be used. Let us prove that the subset $\Omega = \mathbf{S}^n_+$ of the set $V = \mathbf{S}^n$ of symmetric n-by-n matrices, consisting of all *positive semidefinite matrices*, is convex.

Note that doing this via the "nonnegative eigenvalues" definition of positive semidefiniteness would be difficult. Luckily, there is another definition: a matrix $M \in \mathbf{S}^n_+$ is positive semidefinite if and only if $x'Mx \geq 0$ for all $x \in \mathbf{C}^n$. Note that any $x \in \mathbf{C}^n$ defines an affine (actually, a linear) functional $f = f_x : \mathbf{S}^n \to \mathbf{R}$ according to

$$f_x(M) = x'Mx.$$

Hence, $\mathbf{S}^n_+$ is a subset of $\mathbf{S}^n$ defined by some (infinite) set of linear inequalities. According to Lemma 9.1, $\mathbf{S}^n_+$ is a convex set.

9.1.3 Convex Functions

Let $f : \Omega \to \mathbf{R}$ be a function defined on a subset $\Omega \subset V = \mathbf{R}^n$. Function f is called *convex* if the set

$$\Gamma^+_f = \{(v, y) \in \Omega \times \mathbf{R} : \ y \geq f(v)\},$$

is a convex subset of $V \times \mathbf{R}$.

According to this definition, $f : \Omega \to \mathbf{R}$ is convex if and only if the following two conditions hold:

(a) Ω is convex;

(b) the inequality
$$f(cv_1 + (1-c)v_2) \leq cf(v_1) + (1-c)f(v_2)$$
holds for all $v_1, v_2 \in V$, $c \in [0,1]$.

Note that condition (b) has the meaning that any segment connecting two points on the graph of f lies em above the graph of f.

The definition of a convex function does not help much with proving that a given function is convex. The following three statements are of great help in establishing convexity of functions.

Let us call a function $f:\ \Omega \to \mathbf{R}$ defined on a subset Ω of $\mathbf{R}^n$ *twice differentiable* at a point $v_0 \in \Omega$ if there exists a symmetric matrix $W \in \mathbf{S}^n_{\mathbf{R}}$ and a row vector p such that

$$\frac{f(v) - f(v_0) - p(v - v_0) - 0.5(v - v_0)'W(v - v_0)}{\|v - v_0\|^2} \to 0 \ \text{ as } \ v \to v_0,\ v \in \Omega,$$

in which case $p = f'(v_0)$ is called the *first derivative* of f at v_0 and $W = f''(v_0)$ is called the *second derivative* of f at v_0.

Lemma 9.2 *Let $\Omega \subset \mathbf{R}^n$ be a convex subset of $\mathbf{R}^n$. Let $f:\ \Omega \to \mathbf{R}$ be a continuous function which is twice differentiable on the interior of Ω (assumed to be not empty). Then f is convex on Ω if and only if $f''(\omega)$ positive semidefinite in the interior of Ω.*

For example, let Ω be the positive quadrant in $\mathbf{R}^2$, i.e. the set of vectors $[x; y] \in \mathbf{R}^2$ with positive components $x > 0$, $y > 0$. Obviously Ω is convex. Let the function $f:\ \Omega \to \mathbf{R}$ be defined by $f(x, y) = 1/xy$. According to Lemma 9.2, f is convex, because the second derivative

$$W(x, y) = \begin{bmatrix} d^2f/dx^2 & d^2f/dxdy \\ d^2f/dydx & d^2f/dy^2 \end{bmatrix} = \begin{bmatrix} 2/x^3y & 1/x^2y^2 \\ 1/x^2y^2 & 2/xy^3 \end{bmatrix}$$

is positive definite on Ω.

Lemma 9.3 *Let $\Omega \subset V$ be a convex subset of $V = \mathbf{R}^n$. Let P be a set of affine functionals on V such that*

$$f(v) = \sup_{p \in P} p(v) < \infty \quad \forall\ v \in \Omega.$$

Then $f:\ \Omega \to \mathbf{R}$ is a convex function.

To give an example of how Lemma 9.3 can be used, let us prove that the function $f:\ \mathbf{C}^{n,m} \to \mathbf{R}$ defined by $f(M) = \sigma_{\max}(M)$ is convex, where $\mathbf{C}^{n,m}$ denotes the set of all n-by-m matrices with complex entries. Though $\Omega = \mathbf{C}^{n,m}$ is in a simple one-to-one correspondence with $\mathbf{R}^{2nm}$, using Lemma 9.2 to prove convexity of f is essentially impossible: f is not differentiable at many points, and its second derivative, where exists, is cumbersome to calculate. Luckily, from linear algebra we know that

$$\sigma_{\max}(M) = \max\{\mathrm{Re}(p'Mq):\ p \in \mathbf{C}^n,\ q \in \mathbf{C}^m,\ \|p\| = \|q\| = 1\}.$$

Since each individual function $M \mapsto \mathrm{Re}(p'Mq)$ is linear, Lemma 9.3 implies that f is convex.

In addition to Lemma 9.2 and Lemma 9.3, which help establishing convexity “from scratch”, the following statements can be used to derive convexity of one function from convexity of other functions.

Lemma 9.4 *Let V be a vector space, $\Omega \subset V$.*

(a) If $f: \ \Omega \to \mathbf{R}$ and $g: \ \Omega \to \mathbf{R}$ are convex functions then $h: \ \Omega \to \mathbf{R}$ defined by $h(v) = f(v) + g(v)$ is convex as well.

(b) If $f: \ \Omega \to \mathbf{R}$ is a convex function and $c > 0$ is a positive real number then $h: \ \Omega \to \mathbf{R}$ defined by $h(v) = cf(v)$ is convex.

(c) If $f: \ \Omega \to \mathbf{R}$ is a convex function, U is a vector space, and $L: \ U \to V$ is an affine *function, i.e.*

$$L(cu_1 + (1-c)u_2) = cL(u_1) + (1-c)L(u_2) \ \ \forall \ c \in \mathbf{R}, \ u_1, u_2 \in U,$$

then the set

$$L^{-1}(\Omega) = \{u \in U : \ L(u) \in \Omega\}$$

is convex, and the function $f \circ L: \ L^{-1}(\Omega) \to \mathbf{R}$ defined by $(f \circ L)(u) = f(L(u))$ is convex.

For example, let $g: \ \mathbf{S}^3_{\mathbf{R}} \to \mathbf{R}$ be defined on symmetric 2-by-2 matrices by

$$g\left(\begin{bmatrix} x & y \\ y & z \end{bmatrix}\right) = x^2 + y^2 + z^2.$$

To prove that g is convex, note that $g = f \circ L$ where $L: \ \mathbf{S}^3 \to \mathbf{R}^3$ is the affine (actually, linear) function defined by

$$L\left(\begin{bmatrix} x & y \\ y & z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and $f: \ \mathbf{R}^3 \to \mathbf{R}$ is defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x^2 + y^2 + z^2.$$

Lemma 9.2 can be used to establish convexity of f (the second derivative of f turns out to be the identity matrix). According to Lemma 9.4, g is convex as well.

9.1.4 Quasi-Convex Functions

Let $\Omega \subset V$ be a subset of a vector space. A function $f:\ \Omega \to \mathbf{R}$ is called *quasi-convex* if its *level sets*

$$\Omega_\gamma = \{v \in \Omega :\ f(v) < \gamma\}$$

are convex for all γ.

It is easy to prove that any convex function is quasi-convex. However, there are many important *quasi-convex* functions which are not convex. For example, let $\Omega = \{(x,y) : x > 0, y > 0\}$ be the positive quadrant in $\mathbf{R}^2$. The function $f:\ \Omega \to \mathbf{R}$ defined by $f(x,y) = -xy$ is not convex but quasi-convex.

A rather general definition leading to quasi-convex functions is given as follows.

Lemma 9.5 *Let $\Omega \subset V$ be a subset of a vector space. Let $P = \{(p,q)\}$ be a set of pairs of affine functionals $p,q:\ \Omega \to \mathbf{R}$ such that*

(a) inequality $p(v) \geq 0$ holds for all $v \in \Omega$, $(p,q) \in P$;

(b) for any $v \in \Omega$ there exists $(p,q) \in P$ such that $p(v) > 0$;

(c) for every $v \in \Omega$ there exists $\lambda \in \mathbf{R}$ such that $\lambda p(v) \geq q(v)$ for all $(p,q \in P$.

Then the function $f:\ \Omega \to \mathbf{R}$ defined by

$$f(v) = \inf\{\lambda :\ \lambda p(v) \geq q(v)\ \forall\ (p,q) \in P\} \tag{9.1}$$

is quasi-convex.

For example, the *largest generalized eigenvalue* function $f(v) = \lambda_{\max}(\alpha,\beta)$ defined on the set $\Omega = \{v\}$ of pairs $v = (\alpha,\beta)$ of matrices $\alpha,\beta \in \mathbf{S}^n$ such that α is positive semidefinite and $\alpha \neq 0$, is quasi-convex. To prove this, recall that

$$\lambda_{\max}(\alpha,\beta) = \inf\{\lambda :\ \lambda x'\alpha x \geq x'\beta x\ \forall\ x \in \mathbf{C}^n\}.$$

This is a representation of $\lambda_{\max}$ in the form (9.1) with $((p,q) = (p_x,q_x)$ defined by an $x \in \mathbf{C}^n$ according to

$$p_x(v) = x'\alpha x,\ \ q_x(v) = x'\beta x \ \text{ where }\ v = (\alpha,\beta).$$

Since for any $\alpha \geq 0$, $\alpha > 0$ there exists $x \in \mathbf{C}$ such that $x'\alpha x > 0$, Lemma 9.5 implies that $\lambda_{\max}$ is quasi-concave on Ω.

9.2 Standard Convex Optimization Setups

There exists a variety of significantly different tasks commonly referred to as convex optimization problems.

9.2.1 Minimization of a Convex Function

The standard general form of a convex optimization problem is *minimization* $f(v) \to \min$ of a *convex function* $f: \ \Omega \to \mathbf{R}$.

The remarkable feature of such optimization is that for every point $v \in \Omega$ which is *not* a minimum of f and for every number $\gamma \in (\inf(f), f(v))$ there exists a vector u such that $v + tu \in \Omega$ and $f(v + tu) \leq f(v) + t(\gamma - f(v))$ for all $t \in [0, 1]$. (In other words, f is decreasing quickly in the direction u.) In particular, every local minimum of a convex function is its global minimum.

While it is reasonable to expect that convex optimization problems are easier to solve, and reducing a given design setup to a convex optimization is frequently a major research objective, it must be understood clearly that convex optimization problems are useful only when the task of calculating $f(v)$ for a given v (which includes checking that $v \in \Omega$) is not too complicated.

For example, let X be any finite set and let $g: \ X \to \mathbf{R}$ be *any* real-valued function on X. Minimizing g on X can be very tricky when the size of X is large (because there is very little to offer apart from the random search). However, after introducing the vector space V of all functions $v: \ X \to \mathbf{R}$, the convex set Ω can be defined as the set of all *probability distributions* on X, i.e. as the set of all $v \in V$ such that

$$v(x) \geq 0 \ \forall \ x, \quad \sum_{x \in X} v(x) = 1,$$

and $f: \ \Omega \to \mathbf{R}$ can be defined by

$$f(v) = \sum_{x \in X} g(x) v(x).$$

Then f is convex and, formally speaking, minimization of g on X is “equivalent” to minimization of f on Ω, in the sense that the argument of minimum of f is a function $v \in \Omega$ which is non-zero only at those $x \in X$ for which $g(x) = \min(g)$. However, unless some nice simplification takes place, $f(v)$ is “difficult” to evaluate for a particular v (the “brute force” way of doing this involves calculation of $g(x)$ for *all* $x \in X$), this “reduction” to the convex optimization does not make much sense.

9.2.2 Linear Programs

As it follows from Lemma 9.1, a convex set Ω can be defined by a family of linear inequalities. Similarly, according to Lemma 9.3, a convex function can be defined as supremum of a family of affine functions. The problem of finding the minimum of f on Ω when Ω is a subset of $\mathbf{R}^n$ defined by a *finite* family of linear inequalities, i.e.

$$\Omega = \{v \in \mathbf{R}^n : \; a_i' v \le b_i, \; i = 1, \dots, m\}, \tag{9.2}$$

and $f : \; \Omega \to \mathbf{R}$ is defined as supremum of a *finite* family of affine functions,

$$f(v) = \max_{i=1,\dots,k} c_i' v + d_i, \tag{9.3}$$

where a_i, c_i are given vectors in $\mathbf{R}^n$, and b_i, d_i are given real numbers, is referred to as a *linear program.*

In fact, all linear programs defined by (9.2),(9.3) can be reduced to the case when f is a *linear* function, by appending an extra component v_{n+1} to v, so that the new decision variable becomes

$$\bar{v} = \begin{bmatrix} v \\ v_{n+1} \end{bmatrix} \in \mathbf{R}^{n+1},$$

introducing the additional linear inequalities

$$\bar{c}_i' \bar{v} = c_i' v - v_{n+1} \le -d_i,$$

and defining the new objective function $\bar{f}$ by

$$\bar{f}(\bar{v}) = v_{n+1}.$$

Most linear programming optimization engines would work with the setup (9.2),(9.3), where $f(v) = Cv$ is a linear function. The common equivalent notation in this case is

$$Cv \to \min \quad \text{subject to} \quad Av \le B,$$

where a_i' are the rows of A, b_i are the elements of the column vector B, and the inequality $Av \le B$ is understood component-wise.

9.2.3 Semidefinite Programs

A *semidefinite program* is typically defined by an affine function $\alpha : \; \mathbf{R}^n \to \mathbf{S}^N_{\mathbf{R}}$ and a vector $c \in \mathbf{R}^n$, and is formulated as

$$c' v \to \min \quad \text{subject to} \quad \alpha(v) \ge 0. \tag{9.4}$$

Note that in the case when

$$\alpha(v) = \begin{bmatrix} b_1 - a_1'v & & 0 \\ & \ddots & \\ 0 & & b_N - a_N'v \end{bmatrix}$$

is a diagonal matrix valued function, the special semidefinite program becomes a general linear program. Therefore, linear programming is a special case of semidefinite programming.

Since a single matrix inequality $\alpha \geq 0$ represents an infinite number of inequalities $x'\alpha x \geq 0$, semidefinite programs can be used to represent constraints much more efficiently than linear programs. The KYP Lemma explains the special importance of linear matrix inequalities in system analysis and optimization. On the other hand, software for solving general semidefinite programs appears to be not as well developed as in the case of linear programming.

9.2.4 Smooth Convex Optimization

Smooth convex optimization involves minimization of a twice differentiable convex function $f: \Omega \to \mathbf{R}$ on an *open* convex set $\Omega \subset \mathbf{R}^n$ in the situation when $f(v)$ approaches infinity whenever v approaches the boundary of Ω or infinity.

This case can be solved very efficiently using an iterative algorithm which updates its current guess v_t at the minimum in the following way. Let $p_t' = f'(v_t)$, $W_t = f''(v_t) > 0$. Keeping in mind that

$$f(v_t + \delta) \approx \sigma_t(\delta) = f(v_t) + p_t'\delta + 0.5\delta' W_t \delta$$

can be approximated by a quadratic form, let

$$\delta_t = -W_t^{-1} p_t$$

be the argument of minimum of $\sigma_t(\delta)$. Let $\tau = \tau_t$ be the argument of minimum of $f_t(\tau) = f(v_t + \tau\delta)$ (since τ is a scalar, such a minimum is usually easy to find). Then set $v_{t+1} = v_t + \tau_t \delta_t$ and repeat the process.

Actually, non-smooth convex optimization problems (such as linear and semidefinite programs) are frequently solved by reducing them to a sequence of smooth convex optimizations.

9.2.5 Feasibility Search and Quasi-Convex Optimization

Convex feasibility search problems are formulated as the problems of finding an element in a convex set Ω described implicitly by a set of convex constraints. In most situations, it is easy to convert a convex feasibility problem to a convex optimization problem. For example, the problem of finding a $x \in \mathbf{R}^n$ satisfying a finite set of linear inequalities $a_i'x \le b_i$, $i = 1, \dots, N$, can be converted to a linear program

$$y \to \min \quad \text{subject to} \quad a_i'x - y \le b_i, \ (i = 1, \dots, N).$$

If $y = y_0 \le 0$ for some $v_0 = (y_0, x_0)$ satisfying the constraints then $x = x_0$ is a solution of the original feasibility problem. Otherwise, if y is always positive, the feasibility problem has no solution.

In turn, quasi-convex optimization problems can be reduced to convex feasibility search. Consider the problem of minimization of a given quasi-convex function $f : \Omega \to \mathbf{R}$. Assume for simplicity that the values of f are limited to an interval $[f_{\max}, f_{\min}]$. As in the algorithm for H-Infinity optimization, set $\gamma_- = f_{\min}$, $\gamma_+ = f_{\max}$, and repeat the following step until the ratio $(\gamma_+ - \gamma_-)/(f_{\max} - f_{\min})$ becomes small enough: solve the convex feasibility problem of finding $v \in \Omega$ such that $f(v) \le \gamma$ where $\gamma = 0.5(\gamma_- + \gamma_+)$; if such v exists, set $\gamma_- = \gamma$, otherwise set $\gamma_+ = \gamma$.

9.3 Duality in convex optimization

Duality is extremely important for understanding convex optimization. Practically, it delivers a major way of deriving lower bounds in convex minimization problems.

9.3.1 Dual optimization problem and duality gap

According to the remarks made before, a rather general class of convex optimization problems is represented by the setup

$$f(v) = \max_{r \in \mathcal{R}} \{a_r v + b_r\} \to \min \ \text{subject to} \ v \in \Omega = \{v : \max_{k \in \mathcal{K}} \{c_k v + d_k\} \le 0\}, \tag{9.5}$$

where a_r, c_k are given row vectors indexed by $r \in \mathcal{R}$, $k \in \mathcal{K}$ (the sets $\mathcal{R}$, $\mathcal{K}$ are not necessarily finite), b_r, d_k are given real numbers, and v is a column decision vector. When $\mathcal{K}, \mathcal{R}$ are finite sets, (9.5) defines a linear program.

Consider some functions $u : \mathcal{R} \mapsto \mathbf{R}$ and $q : \mathcal{K} \mapsto \mathbf{R}$ which assign real numbers to the indexes, in such a way that only a countable set of values $u(r) = u_r$, $q(k) = q_k$ is positive, and

$$u_k \ge 0, \ \sum_k u_k = 1, \quad q_r \ge 0, \ \sum_r q_r \le 1. \tag{9.6}$$

Obviously,

$$f(v) \geq \sum_r u_r(a_r v + b_r),$$

and

$$\sum_k q_k(c_k v + d_k) \leq 0 \quad \forall\ v \in \Omega.$$

Hence

$$f(v) \geq \sum_r u_r b_r + \sum_k q_k d_k \quad \forall\ v \in \Omega$$

whenever

$$\sum_r u_r a_r + \sum_k q_k c_k = 0, \tag{9.7}$$

in which case

$$\sum_r u_r b_r + \sum_k q_k d_k \tag{9.8}$$

is a lower bound for the minimum in (9.5). Trying to maximize the lower bound leads to the task of maximizing (9.8) subject to (9.6),(9.7). This task, a convex optimization problem itself, is called *dual* with respect to (9.5).

The key property of the dual problem is that its maximum (more precisely, supremum, since the maximum is not necessarily achievable) equals the minimum (infimum) in the original optimization problem (9.5).

9.3.2 The Hahn-Banach Theorem

The basis for all convex duality proofs is the fundamental *Hahn-Banach Theorem*. The theorem can be formulated in two forms: geometric (easier to understand) and functional (easier to prove).

By definition, an element v_0 of a real vector space V is called an *interior point* of a subset $\Omega \subset V$ if for every $v \in V$ there exists $\epsilon = \epsilon_v > 0$ such that $v_0 + tv \in \Omega$ for all $|t| < \epsilon_v$.

Theorem 9.1 *Let Ω is a convex subset of a real vector space V such that 0 is an interior point of Ω. If $v_0 \in V$ is not an interior point of Ω then there exists a linear function $L: V \mapsto \mathbf{R}$, $L \not\equiv 0$, such that*

$$L(v_0) \geq \sup_{v \in \Omega} L(v).$$

In other words, a point not strictly inside a convex set can be separated from the convex set by a hyperplane.

To give an alternative formulation of the Hahn-Banach Theorem, remember that a non-negative function $q: V \mapsto \mathbf{R}$ defined on a real vector space V is called a *semi-norm* if it is convex and positively homogeneous (i.e. $p(av) = ap(v)$ for all $a \geq 0$, $v \in V$).

Theorem 9.2 *Let $q: V \mapsto \mathbf{R}$ be a semi-norm on a real vector space V. Let V_0 be a linear subspace of V, and $h_0: V_0 \mapsto \mathbf{R}$ be a linear function such that $q(v) \geq h_0(v)$ for all $v \in V_0$. Then there exists a linear function $h: V \mapsto \mathbf{R}$ such that $h(v) = h_0(v)$ for all $v \in V_0$, and $h(v) \leq q(v)$ for all $v \in V$.*

To relate the two formulations, define $q(v)$ as the *Minkovski' functional* of Ω:

$$q(v) = \inf\{t > 0 : \; t^{-1}v \in \Omega\},$$

and set

$$V_0 = \{tv_0 : \; t \in \mathbf{R}\}, \quad h_0(tv_0) = t.$$

9.3.3 Duality gap for linear programs

To demonstrate utility of the Hahn-Banach theorem, let us use it to prove the "zero duality gap" statement for linear programs.

Theorem 9.3 *Let A, B, C be real matrices of dimensions n-by-m, n-by-1, and 1-by-m respectively. Assume that there exists $v_0 \in \mathbf{R}^m$ such that $Av_0 < B$. Then*

$$\sup\{Cv : \; v \in \mathbf{R}^m, \; Av \leq B\} = \inf\{B'p : \; p \in \mathbf{R}^n, \; A'p = C', \; p \geq 0\}. \tag{9.9}$$

The inequalities $Av \leq B$, $Av_0 < B$, and $p \geq 0$ in (9.9) are understood component-wise. Note also that inf over an empty set equals *plus infinity*. This can be explained by the fact that inf is the *maximal lower bound* of a set. Since every number is a lower bound for an empty set, its infimum equals $+\infty$. Theorem 9.3 remains valid when there exist no $p \geq 0$ such that $A'p = C'$, in which case it claims that inequality $Av \leq B$ has infinitely many solutions, among which Cv can be made arbitrarily small.

Proof The inequality

$$\sup\{Cv : \; v \in \mathbf{R}^m, \; Av \leq B\} \leq \inf\{B'p : \; p \in \mathbf{R}^n, \; A'p = C', \; p \geq 0\}$$

is straightforward: multiplying $Av \leq B$ by $p' \geq 0$ on the left yields $p'Av \leq B'p$; when $A'p = C'$, this yields $Cv \leq B'p$.

The proof of the inverse inequality

$$\sup\{Cv:\ v \in \mathbf{R}^m,\ Av \le B\} \ge \inf\{B'p:\ p \in \mathbf{R}^n,\ A'p = C',\ p \ge 0\}$$

relies on the Hahn-Banach theorem.

Let y be an upper bound for Cv subject to $Av \le B$. If $y = \infty$ then, according to the already proven inequality, there exist no $p \ge 0$ such that $A'p = C'$, and hence the desired equality holds.

If $y < \infty$, let e denote the n-by-1 vector with all entries equal to 1. Consider the set

$$\Omega = \left\{ x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} Cv - \delta + 1 \\ e - Av - \Delta \end{bmatrix} \in \mathbf{R}^{n+1}:\ \Delta > 0,\ \delta > 0 \right\}.$$

Then

(a) Ω is a convex set (as a linear transformation image of a set defined by linear inequalities);

(b) zero is an interior point of Ω (because it contains the open cube $|x_i| < 1$, which can be seen by setting $v = 0$);

(c) vector $[y + 1; e - B]$ does not belong to Ω (otherwise $Av + \Delta = B$ and $Cv - \delta = y$, which contradicts the assumption that $Cv \le y$ whenever $Av \le B$).

According to the Hahn-Banach Theorem, this means that there exists a non-zero linear functional

$$L \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} = L_0 x_0 + \bar{L}'\bar{x},$$

where $L_0 \in \mathbf{R}$, $\bar{L} \in \mathbf{R}^n$, defined on $\mathbf{R}^{n+1}$, such that

$$L_0(Cv - \delta + 1) + \bar{L}'(e - Av - \Delta) \le L_0(y + 1) + \bar{L}'(e - B)\ \ \forall\, \Delta > 0, \delta > 0, v. \quad (9.10)$$

Looking separately at the coefficients at v, δ, Δ and at the constant term in (9.10) implies

$$L_0 C = \bar{L}'A,\ L_0 \ge 0,\ \bar{L} \ge 0,\ L_0 y \ge \bar{L}'B. \quad (9.11)$$

Note that L_0 cannot be equal to zero: otherwise $\bar{L}'A = 0$ and $\bar{L}'B \ge 0$, which, after multiplying $Av_0 < B$ by $\bar{L} \ge 0$, $\bar{L} \ne 0$ yields a contradiction:

$$0 = \bar{L}'Av_0 < \bar{L}'B \le L_0 y = 0.$$

If $L_0 > 0$ then for

$$p = \bar{L}/L_0$$

conditions (9.11) imply

$$A'p = C', \quad p \geq 0, \quad B'p \leq y.$$

■

9.4 Algorithms for convex optimization

This section describes some algorithms and software which can be used for solving convex optimization problems.

9.4.1 Method of ellipsoids

Let $f: \ \Omega \mapsto \mathbf{R}$ be a a quasi-convex function defined on a bounded set $\Omega \subset \mathbf{R}^n$.

While no specific analytical description of f and Ω will be used, it is assumed that, for a given $v \in \mathbf{R}^n$, it is possible to check whether condition $v \in \Omega$ is satisfied. The function performing this task is called *feasibility oracle*. It will be assumed that, in addition to verifying condition $v \in \Omega$, the oracle calculates coefficients p_v, q_v of an affine functional $h_v(x) = p_v x - q_v$ which separates v from Ω in case $v \notin \Omega$, i.e.

$$p_v x \geq q_v \ \forall \ x \in \Omega, \ p_v v \leq q_v, \quad \text{if} \ \ v \notin \Omega,$$

and separates v from the level set

$$\Omega_v = \{x \in \Omega : \ f(x) \leq f(v)\}$$

when $v \in \Omega$, i.e.

$$p_v(x - v) \geq f(v) - f(x) \ \forall \ x \in \Omega, \ p_v v = f(v), \quad \text{if} \ \ v \in \Omega.$$

Also, the following a-priori assumptions will be made about f and Ω:

(a) Ω is a bounded set, i.e. $\|v\| \leq R$ for all $v \in \Omega$, where $R > 0$ is a given constant.

(b) f is bounded from below on Ω, i.e.

$$f_o = \inf\{f(v) : \ v \in \Omega\} > -\infty,$$

and the level set Ω_v does not shrink too rapidly as $f(v)$ approaches f_o, i.e. there exist constants $c, m > 0$ such that the volume of Ω_v is not smaller than $c(f(v) - f_o)^m$ for all $v \in \Omega$.

Uder these assumptions, the *ellipsoids algorithm* allows one to find an element $v_\epsilon \in \Omega$ such that $f(v_\epsilon) - f_o < \epsilon$ using $O(m\log(\epsilon)/\log(R))$ calls to the oracle, and $O(n^2 m\log(\epsilon)/log(R))$ real arithmetic operations.

At step k, the standard ellipsoid algorithm updates the current suboptimality level γ_k, an element $v_k \in \mathbf{R}^n$ such that $f(v_k) \leq \gamma_k$ (where $f(v)$ for $v \notin \Omega$ is defined as $+\infty$), and an ellipsoid

$$E_k = \{x \in \mathbf{R}^n : \ (x - x_k)' Q_k^{-1} (x - x_k) \leq 1\}$$

which contains all $x \in \Omega$ such that $f(x) < \gamma_k$. At the beginning, the parameters are initialized at

$$\gamma_0 = \infty, \quad v_0 = x_0 = 0, \quad Q_0 = R^2 I_n.$$

To get γ_k, v_k, E_k from γ_{k-1}, v_{k-1}, E_{k-1}, one applies the feasibility oracle to x_{k-1}, and defines E_k as the minimal volume ellipsoid containing the intersection of E_{k-1} with the hyperplane

$$p_k x + q_k = h_k(x) = h_v(x) \geq 0$$

produced by the oracle, which yields explicit formulae for x_k, Q_k as functions of $x_{k-1}, Q_{k-1}, p_k, q_k$. In addition, if $x_{k-1} \in \Omega$ and $f(x_{k-1}) < \gamma_{k-1}$, v_k is upgraded as x_{k-1}, and γ_k is upgraded as $f(x_{k-1}) = q_k$, otherwise $\gamma_k = \gamma_{k-1}$ and $v_k = v_{k-1}$.

It can be shown that the volume of E_k is never larger than $(1 - 1/(2n))^k$ times the volume of E_0, which proves the convergence properties of the algorithm. In must be noted that, despite having remarkable *provable* convergence properties, most practical implementations of the ellipsoids algorithm turn out to be inferior to the alternatives, such as the interior point method.

9.4.2 Linear programming

The optimization toolbox of MATLAB provides function *linprog.m* for solving linear programs. The simplest call format is

```
v=linprog(C',A,B)
```

to solve the problem of *minimizing* Cv subject to $Av \leq B$.

My past experience with this function is not very positive: it starts failing already for very moderately sized tasks. An alternative (and also a *free* option) is the SeDuMi package, which can be downloaded from

```
http://fewcal.kub.nl/sturm/software/sedumi.html
```

When SeDuMi is installed, it can be used to solve simultaneously the dual linear programs

$$Cx \to \max \quad \text{subject to} \quad B - Ax \geq 0$$

and

$$B'p \to \min \quad \text{subject to} \quad A'p = C', \ p \geq 0$$

by calling

```
[p,x]=sedumi(A',C',B);
```

Actually, the indended use of SeDuMi is solving semidefinite programs, which can be achieved by changing the interpretation of the ≥ 0 condition (set by the fourth argument of `sedumi`). In general, inequality $z \geq 0$ will be interpreted as $z \in \mathcal{K}$, where $\mathcal{K}$ is a *self-dual cone*. Practically speaking, by saying that $z \in \mathcal{K}$ one can specify that certain elements of vector z must form positive semidefinite matrices, instead of requiring the elements to be non-negative.

Note that both `linprog.m` and `sedumi.m` require the primal and dual optimization problems to be *strictly feasible* (i.e. inequalities $Ax < B$ and $p > 0$ subject to $A'p = C'$ must have solutions). One can argue that a well formulated convex optimization problem should satisfy this condition anyway.

9.4.3 Semidefinite programming

While SeDuMi is easy to apply for solving *some* semidefinite programs, it is frequently inconvenient for situations related to control systems analysis and design. A major need there is to be able to define matrix equalities or inequalities in a "block format", such as in the case of a Lyapunov inequality

$$AP + PA' = Y \geq 0, \ P > 0$$

where A is a given square matrix, and $P = P'$, $Y = Y'$ are matrix decision parameters. The LMI Control Toolbox of MATLAB provides interface commands for defining linear matrix inequalities in a block matrix format. However, this interface itself is quite scriptic, and hence is not easy to work with.

The package IQCbeta, freely available from

`http://www.math.kth.se/~cykao/`

and already installed on Athena, helps to cut significantly the coding effort when solving semidefinite programs.

To use IQCbeta, put the content of

`http://web.mit.edu/6.245/www/images/startup6245.m`

into your `startup.m` file (should be in your `~/matlab/` directory).

Here is an example of a function which will minimize the largest eigenvalue of $PA+A'P$ where A is a given matrix, and P is the symmetric matrix decision variable satisfying $0 \le P \le I$.

```
function P=example_sdp_lyapunov(A)
% function P=example_sdp_lyapunov(A)
%
% demonstrates the use of IQCbeta by finding P=P' which minimizes
% the largest eigenvalue of PA+A'P subject to 0<=P<=I

n=size(A,1);        % problem dimension
abst_init_lmi;      % initialize the LMI solving environment
p=symmetric(n);     % p is n-by-n symmetric matrix decision variable
y=symmetric;        % y is a scalar decision variable
p>0;                % define the matrix inequalities
p<eye(n);
p*A+A'*p<y*II(n);
lmi_mincx_tbx(y);   % call the SDP optimization engine
P=value(p);         % get value of the optimal p
```

9.5 Convexifivcation of model reduction problems

A typical model reduction problem is not convex in an "obvious" way. This is mainly due to the fact that the set of all rational functions of a given order is not convex. For example, the arithmetic mean of n first order transfer functions $F_k(s) = 1/(s+k)$, where $k = 1, 2, \ldots, n$, has order n, not 1. Nevertheless, there are several cases when an optimal model reduction problem can be "convexified" by introducing a new set of parameters.

9.5.1 Quasi-convexity of real rational approximations

Consider the task of finding the best uniform approximation of a given non-negative function $F: [0,\infty) \mapsto [0,\infty)$ by a positive rational function

$$\hat{F}(\theta) = \frac{b(\theta)}{a(\theta)} = \frac{b_0 + b_1\theta + \cdots + b_m\theta^m}{a_0 + a_1\theta + \cdots + a_{m-1}\theta^{m-1} + \theta^m}$$

of a given order m. The term "uniform" means that the functional

$$\mathrm{dist}\{F, \hat{F}\} = \sup_{\theta\in[0,\infty)} |F(\theta) - \hat{F}(\theta)|$$

is to be minimized. It can be assumed without loss of generality that $a(\theta) > 0$ and $b(\theta) > 0$ for $\theta \in [0,\infty)$.

Let

$$v = [b_0; b_1; \ldots; b_m; a_0; a_1; \ldots; a_{m-1}] \in \mathbf{R}^{2m+1}$$

be the vector of decision parameters in this "model reduction" problem. Let $\Omega \subset \mathbf{R}^{2m+1}$ denote the set of all v such that the resulting polynomials $a = a(\theta)$ and $b = b(\theta)$ are positive for $t \in [0,\infty)$. The approximation quality measure $m(\hat{F}) = \mathrm{dist}(F, \hat{F})$ defines a function $f: \Omega \mapsto \mathbf{R}$. It turns out that f is a *quasi-convex* function.

Indeed, the set $\Omega = \{v\}$ is defined by an (infinite) family of linear inequalities

$$p^a_\theta v + q^a_\theta > 0, \quad q^b_\theta v > 0,$$

with respect to $v \in \mathbf{R}^{2m+1}$, where

$$p^a_\theta = \begin{bmatrix} 0 \\ \theta_{m-1} \end{bmatrix}, \quad q^a_\theta = \theta^m, \quad p^b_\theta = \begin{bmatrix} \theta_m \\ 0 \end{bmatrix}, \quad \theta_k = \begin{bmatrix} 1 \\ \theta \\ \vdots \\ \theta^k \end{bmatrix},$$

parameterized by $\theta \in [0,\infty)$. Hence Ω is a *convex* set.

While it can be shown that the functional $f: \Omega \mapsto \mathbf{R}$ is *not* convex, its level sets

$$\Omega^\gamma = \{v \in \Omega : \ f(v) < \gamma\}$$

can be proven to be convex. Indeed, Ω^γ is the subset of Ω defined by the inequalities

$$p^{\pm}_{\theta,\gamma} v + q^{\pm}_{\theta,\gamma} > 0,$$

where

$$p^{+}_{\theta,\gamma} = \begin{bmatrix} \theta_m \\ (\gamma - F(\theta))\theta_{m-1} \end{bmatrix}, \quad q^{+}_{\theta,\gamma} = (\gamma - F(\theta))\theta^m,$$

$$p^{-}_{\theta,\gamma} = \begin{bmatrix} -\theta_m \\ (\gamma + F(\theta))\theta_{m-1} \end{bmatrix}, \quad q^{+}_{\theta,\gamma} = (\gamma + F(\theta))\theta^m,$$

and hence Ω^γ is convex for all γ.

Thanks to the quasi-convexity feature, the task of minimizing f on Ω can be solved numerically by using the ellipsoid algorithm. Alternatively, this can be done by combining semidefinite programming with a binary search, or, somewhat less efficiently, by combining linear programming with a binary search. These options are explored in the following subsections.

9.5.2 Ellipsoid algorithm in real rational approximations

As it was mentioned earlier, in order to apply the ellipsoid algorithm to the optimal real rational approximation problem, one has to provide the following three items:

(a) a feasibility oracle;

(b) an ellipsoid containing Ω;

(c) an assurance that the volume of Ω^γ does not decrease too rapidly as γ approaches its minimum.

It is easy to see that condition (b) cannot be satisfied without changing the parameterization again. Indeed, the polynomials

$$b_r(\theta) \equiv 1, \quad a_r(\theta) = (1 + r\theta)^m$$

define an element $v \in \Omega$ for all $r > 0$, and the coefficients of a are unbounded as $r \to \infty$ despite the fact that $\text{dist}(F, b_r/a_r)$ is bounded. Hence some level sets of f are unbounded sets.

A resolution of this problem (caused, in fact, by using coefficients of polynomials as parameters, ignoring the fact that the powers θ^k of $\theta \in [0, \infty)$ do not form an orthogonal basis) lies in re-defining the vector of decision parameters. Consider the substitution

$$\theta = \omega_0^2 \tan^2(u/2), \quad u \in (-\pi, \pi),$$

where $\omega_0 > 0$ is a fixed positive number. Then

$$\frac{b(\theta)}{a(\theta)} = \frac{\tilde{b}(u)}{\tilde{a}(u)},$$

where

$$\tilde{b}(u) = \sum_{k=0}^{m} b_k \omega_0^{2m} \cos^{2m-2k}(u/2) \sin^{2m}(u/2),$$

$$\tilde{a}(u) = \sum_{k=0}^{m} a_k \omega_0^{2m} \cos^{2m-2k}(u/2) \sin^{2m}(u/2),$$

and $a_m = 1$. By assumption, $\tilde{a}(u) > 0$ for $u \neq (2n+1)\pi$. Since $a_m = 1$, $\tilde{a}(u) > 0$ for $u = (2n+1)\pi$ as well. Since $\sin^{2k}(u/2)\cos^{2m-2k}(u)$ can be represented as linear combinations of $\cos(nu)$ with $n = 0, 1, \ldots, k$, $\tilde{a}$ and $\tilde{b}$ are actually *trigonometric polynomials*

$$\tilde{a}(u) = \sum_{k=0}^{m} \tilde{a}_k \cos(ku), \quad \tilde{b}(u) = \sum_{k=0}^{m} \tilde{b}_k \cos(ku).$$

In terms of $\tilde{a}, \tilde{b}$, the original problem can be reformulated as the task of minimizing

$$\tilde{f} = \sup_{u \in \mathbf{R}} |\tilde{F}(u) - \tilde{b}(u)/\tilde{a}(u)|,$$

where

$$\tilde{F}(u) = F(\omega_0^2 \tan^2(u/2)),$$

subject to the constraints

$$\tilde{a}(u) > 0, \quad \tilde{b}(u) > 0 \;\; \forall\; u \in \mathbf{R}.$$

Since

$$\tilde{a}_0 = \frac{1}{\pi} \int_0^{\pi} \tilde{a}(u) du > 0,$$

and dividing all coefficients of $\tilde{a}, \tilde{b}$ by the same positive number does not change $\tilde{b}/\tilde{a}$, one can assume that $\tilde{a}_0 = 1$.

Now $\tilde{f} = \tilde{f}(\tilde{v})$ is a quasi-convex function defined over the set $\tilde{\Omega}$ of all vectors

$$\tilde{v} = [\tilde{b}_0; \tilde{b}_1; \ldots; \tilde{b}_m; \tilde{a}_0; \tilde{a}_1; \ldots; \tilde{a}_{m-1}] \in \mathbf{R}^{2m+1}$$

such that

$$1 + \sum_{k=1}^{m} \tilde{a}_k \cos(ku) > 0, \quad \sum_{k=0}^{m} \tilde{b}_k \cos(ku) > 0.$$

Lemma 9.6 *If F is uniformly bounded on $[0, \infty)$, all level sets of $\tilde{F}$ are bounded as well. More precisely,*

$$|a_k| < 2, \quad |b_k| < 2(1 + 2m)(\gamma + \|F\|_\infty)$$

for all $\tilde{v} \in \tilde{\Omega}^\gamma$.

Proof Since

$$1 \pm \tilde{a}_k/2 = \frac{1}{\pi} \int_0^\pi (1 \pm \cos(ku)\tilde{a}(u) du \, 0$$

for $k = 1, \ldots, m$, it follows that $|\tilde{a}_k| < 2$ for all k. Hence

$$|\tilde{a}(u)| \leq 1 + \sum_{k=1}^{m} |\tilde{a}_k| < 1 + 2m.$$

Since $|\tilde{F}(u) - \tilde{b}(u)/\tilde{a}(u)| < \gamma$ for $\tilde{v} \in \tilde{\Omega}^\gamma$, we have

$$|\tilde{b}(u)| \leq |\tilde{F}(u)| \cdot |\tilde{a}(u)| + \gamma |\tilde{a}(u)| < (1 + 2m)(\gamma + \|F\|_\infty).$$

Finally, the bound for $|\tilde{b}_k|$ follows from

$$\tilde{b}_k = \frac{2}{\pi} \int_0^\pi \cos(ku)\tilde{b}(u) du, \; (k > 0), \quad \tilde{b}_0 = \frac{1}{\pi} \int_0^\pi \tilde{b}(u) du.$$

■

Lemma 9.6 provides an initial bounding ellipsoid for the optimal real rational approximation problem. In addition, a feasibility oracle can be introduced under the assumption that $F = F(\theta)$ is a rational function without poles on the non-negative real axis. The oracle is based on a procedure of finding maximum (over $u \in \mathbf{R}$) of a given ratio

$$g(u) = p(u)/q(u)$$

of two trigonometric polynomials

$$p(u) = p_0 + \sum_{k=0}^{n} p_k \cos(ku) = p_0 + \frac{1}{2}\sum_{k=1}^{n} p_k(z^k + z^{-k}),$$

$$q(u) = 1 + \sum_{k=0}^{n} q_k \cos(ku) = q_0 + \frac{1}{2}\sum_{k=1}^{n} q_k(z^k + z^{-k}),$$

where $z = e^{ju}$, and it is assumed that $q(u) > 0$ for all u. Since the derivative of g is given by

$$\dot{g}(u) = \frac{\dot{p}(u)q(u) - p(u)\dot{q}(u)}{q(u)^2},$$

and

$$h(z) = z^{2n}(\dot{p}(u)q(u) - p(u)\dot{q}(u)) \quad (z = e^{ju})$$

is a polynomial of degree not larger than $4n$ with respect to z, the maximum of g can be found by calculating all roots z_i of h such that $|z_i| = 1$, i.e. $z_i = e^{ju_i}$, and then evaluating g at all u_i.

Finally, to show that the level sets $\tilde{\Omega}^{\gamma}$ have sufficiently large volume as γ converges to the infimum of $\tilde{f}$ on $\tilde{W}$, begin with proving existence of an optimal pair $\tilde{b}^*, \tilde{a}^*$ of trigonometric polynomials, for which $\tilde{a}^*(u) > 0$ and $\tilde{b}^*(u) \geq 0$ for all u. Indeed, consider an optimizing sequence of pairs $(\tilde{a}, \tilde{b})$, and use boundedness to show existence of a converging subsequence. For the limit pair $(\tilde{a}, \tilde{b})$, one must have $\tilde{a}(u) \geq 0$ and $\tilde{b} \geq 0$. However, since $\tilde{b}(u)/\tilde{a}(u)$ is bounded, all solutions of equation $\tilde{a}(u) = 0$ must have an even order, and must be also solutions of $\tilde{b}(u) = 0$ of a not lesser order. Hence, all zeros of $a = a(u)$ can be cancelled with the corresponding zeros of $\tilde{b}$, which leads to an optimal $a^*(u) > 0$.

For a given optimal pair $(\tilde{a}^*, \tilde{b}^*)$ such that

$$\tilde{a}^*(u) \geq \epsilon > 0$$

and

$$|\tilde{F}(u) - \tilde{b}^*(u)/\tilde{a}^*(u)| \leq \gamma^* \quad \forall\ u,$$

perturbing $\tilde{a}_k^*$ and $\tilde{b}_k^*$ with $k = 1, \dots, m$ by less than $\delta\epsilon/m)$, and adding a number between $\delta\epsilon$ and $2\delta\epsilon$ to $\tilde{b}_0^*$ yields a pair $\tilde{b}, \tilde{a}$ which satisfy all constraints and satisfy

$$|\tilde{F}(u) - \tilde{b}(u)/\tilde{a}(u)| \leq \gamma^* + \delta \quad \forall\ u.$$

Since the volume of the perturbation set is $O(\epsilon^{2m+1}\delta^{2m+1}m^{-m-1})$, this proves the desired lower bound for the level set volume.

9.5.3 A relaxation for H-Infinity optimal model reduction

One of the most desirable optimization objectives in model reduction of stable LTI systems is minimization of weighted H-Infinity norm of the frequency domain approximation error

$$\|W(G-\hat{G})\|_\infty \to \min: \ \hat{G}(s)=\frac{p(s)}{q(s)}, \ \ q(s)\neq 0 \text{ for } \mathrm{Re(s)}\geq 0, \ \deg(q)=m, \ \deg(p)\leq m, \tag{9.12}$$

where G, W are given stable transfer functions, m is a given positive integer, and p, q are real polynomials to be optimized.

At the moment, no satisfactory solution of problem (9.12) is available. In particular, the problem is not convex with respect to many reasonable choices of decision variables studied in the past. Nevertheless, there exist a way of *relaxing* the original setup, by replacing it with a convex optimization problem which has a broader choice of possible $\hat{G}$. The minimum cost in the relaxed optimization problem provides a lower bound for the minimum in the original weighted H-Infinity optimal model reduction. In addition, the argument of the minimum in the relaxed problem can be used to find a suboptimal (though not optimal) solution in (9.12).

The main idea of the relaxation is common in the theory of model reduction: it allows an *anti-stable* component $\hat{G}_-$ in $\hat{G}(s)=\hat{G}_++\hat{G}_-$ (here *anti-stable* means "strictly proper with no poles in the *left* half plane"). Accordingly, the weighted H-Infinity norm should be "replaced" by the weighted L-Infinity norm (same thing, just extended to non-stable transfer functions). The benefit of adding an anti-stable component may appear to be a little bit unusual: after all, adding an anti-stable transfer function to a stable one *always* increases the overall L2 norm:

$$\int_{-\infty}^{\infty}|G(j\omega)-\hat{G}_+(j\omega)|^2d\omega \leq \int_{-\infty}^{\infty}|G(j\omega)-\hat{G}_+(j\omega)-\hat{G}_-(j\omega)|^2d\omega$$

whenever $G, \hat{G}_+$ are stable and $\hat{G}_-$ is anti-stable. Nevertheless, adding an anti-stable transfer function to a stable transfer function can reduce the overall L-Infinity norm, as in

$$\frac{81}{128}=\left\|\frac{1}{s+1}+\frac{1}{2}\frac{1}{s-1}\right\|_\infty^2 < \left\|\frac{1}{s+1}\right\|_\infty^2=1.$$

The relaxed formulation allows $\hat{G}(s)$ to be of the form

$$\hat{G}(s)=\frac{p(s)}{q(s)}+\frac{r(s)}{q(-s)}, \tag{9.13}$$

where p, q, r are real polynomials, q has degree m and is a Hurwitz polynomial (all roots have negative real part), q has degree not larger than m, and the degree of r is less than m. It turns out that, after a re-parameterization, the objective $\|W(G-\hat{G})\|_\infty$ becomes quasi-convex, which makes it possible to apply standard convex optimization algorithms in its minimization.

9.5.4 Convexifying re-parameterization of the relaxed problem

It turns out that the coefficients of p, q, r in (9.13) do not constitute a convenient set of parameters for minimizing $\|W(G-\hat{G})\|_\infty$. The following lemma provides an alternative set of decision parameters.

Lemma 9.7 *Let $m > 0$ be a given positive integer. Let Ω^m_{qpr} be the set of all triplets (q, p, r) of real polynomials*

$$\begin{aligned} q(s) &= s^m + q_{m-1}s^{m-1} + \cdots + q_1 s + q_0, \\ p(s) &= p_m s^m + p_{m-1}s^{m-1} + \cdots + p_1 s + p_0, \\ r(s) &= r_{m-1}s^{m-1} + \ldots r_1 s + r_0, \end{aligned}$$

satisfying the condition

$$q(s) \neq 0 \quad \text{for} \quad \mathrm{Re}(s) \geq 0. \tag{9.14}$$

Let Ω^m_{abc} be the set of all triplets (a, b, c) of real polynomials

$$\begin{aligned} a(\theta) &= \theta^m + a_{m-1}\theta^{m-1} + \cdots + a_1\theta + a_0, \\ b(\theta) &= b_m\theta^m + b_{m-1}\theta^{m-1} + \cdots + b_1\theta + b_0, \\ c(\theta) &= c_{m-1}\theta^{m-1} + \cdots + c_1\theta + c_0, \end{aligned}$$

satisfying the condition

$$a(\theta) > 0 \quad \text{for} \quad \theta \geq 0. \tag{9.15}$$

There is a one-to-one correspondence $\tau_m : \ \Omega^m_{qpr} \mapsto \Omega^m_{abc}$ between the sets Ω^m_{qpr} and Ω^m_{abc} such that

$$\hat{G}(j\omega) = \frac{p(j\omega)}{q(j\omega)} + \frac{r(j\omega)}{q(-j\omega)} = \frac{b(\omega^2) + j\omega c(\omega^2)}{a(\omega^2)} \quad \forall\ \omega \in \mathbf{R} \tag{9.16}$$

for $(a, b, c) = \tau_m(q, p, r)$. Given $(q, p, r) \in \Omega^m_{qpr}$, the corresponding $(a, b, c) = \tau_m(q, p, r) \in \Omega^m_{abc}$ are defined by

$$\begin{aligned} a(\omega^2) &= q(j\omega)q(-j\omega), \\ b(\omega^2) &= \frac{1}{2}[p(j\omega)q(-j\omega) + p(-j\omega)q(j\omega) + r(j\omega)q(j\omega) + r(-j\omega)q(-j\omega)], \\ c(\omega^2) &= \frac{1}{2j\omega}[p(j\omega)q(-j\omega) - p(-j\omega)q(j\omega) + r(j\omega)q(j\omega) - r(-j\omega)q(-j\omega)]. \end{aligned}$$

Given $(a,b,c)\in\Omega^m_{abc}$, the polynomial q of the corresponding $(q,p,r)=\tau_m^{-1}(a,b,c)\in\Omega^m_{qpr}$ is defined by

$$q(s)=\prod_{a(-s_k^2)=0,\ \mathrm{Re}(s_k)<0}(s-s_k), \tag{9.17}$$

where s_k are all zeros of $a(-s^2)$ with negative real part (multiplicity counts), and p,r are found as the unique *solution of the polynomial equation*

$$p(s)q(-s)+r(s)q(s)=b(-s^2)+sc(-s^2),\quad \deg(p)\le m,\ \deg(q)<m. \tag{9.18}$$

Proof It is easy to see that, in order to satisfy (9.16), polynomials a,b,c have to be defined by $(a,b,c)=\tau_m(q,p,r)$. Let us show that, given $(a,b,c)\in\Omega^m_{abc}$, there is a unique triplet $(q,p,r)\in\Omega^m_{qpr}$ satisfying (9.16), and that this triplet is uniquely defined by (9.17),(9.18).

Indeed, since $a(\omega^2)=q(j\omega)q(-j\omega)$, we have $a(-s^2)=q(s)q(-s)$ for s on the imaginary axis, hence (since a,q are polynomials) $a(-s^2)=q(s)q(-s)$ for all $s\in\mathbf{C}$. Hence all roots of q must be stable roots of $a(-s^2)$. Since $a(-s^2)$ is invariant with respect to replacing s by $-s$, and is not zero on the imaginary axis, $a(-s^2)$ has *exactly* m stable roots (counting multiplicity), which proves (9.17). Now, due to (9.16), identity (9.18) holds for s on the imaginary axis. Hence (since this is a polynomial identity), (9.18) holds for all s. To show existence and uniqueness of solution of (9.18) with respect to p,r, note that the linear function

$$(p,r)\mapsto\delta:\quad \delta(s)=p(s)q(-s)+r(s)q(s)$$

maps a $2m+1$-dimensional vector space (of pairs of real polynomials p,r of degrees not exceeding m and $m-1$ respectively) into a $2m+1$-dimensional vector space (of real polynomials δ of degree not exceeding $2m$).

Hence, to prove existence and uniqueness of solutions, it is sufficient to show that the only pair (p,r) mapped into zero is is $p\equiv 0$, $r\equiv 0$. Indeed, if $\delta\equiv 0$ then

$$r(s)q(s)=p(s)q(-s).$$

Since q has no roots with positive real part, and $q(-s)$ has m roots with positive real part, $r(s)$ must have m roots with positive real part, which implies $r\equiv 0$, as $\deg(r)<m$.
■

9.5.5 Quasi-convexity with respect to a,b,c.

According to Lemma 9.7, minimization of $\|W(G-\hat{G})\|_\infty$ subject to (9.13), where $(q,p,r)\in\Omega^m_{qpr}$ is equivalent to the optimization problem

$$\Phi(a,b,c)=\sup_{\omega\in\mathbf{R}}\left|W(j\omega)\left(G(j\omega)-\frac{b(\omega^2)+j\omega c(\omega^2)}{a(\omega^2)}\right)\right|\to\min,\quad (a,b,c)\in\Omega^m_{abc}. \tag{9.19}$$

The following lemma states quasi-convexity of Φ, thus opening the way torward efficient algorithms for solving (9.19).

Lemma 9.8 *Consider the representation of triplets* $(a, b, c) \in \Omega^m_{abc}$ *as vectors from* $v \in \mathbf{R}^{3m+1}$*, according to*

$$v = [a_{m-1}; \ldots; a_0; b_m; b_{m-1}; \ldots; b_0; c_{m-1}; \ldots; c_0].$$

Then all level sets

$$\Omega^{m,\gamma}_{abc} = \{(a, b, c) \in \Omega^m_{abc} : \ \Phi(a, b, c) < \gamma\}$$

are convex.

Proof $\Omega^{m,\gamma}_{abc}$ is defined by the (infinite) family of inequalities

$$a(\omega^2) > 0, \tag{9.20}$$

$$\left| W(j\omega) \left(G(j\omega) - \frac{b(\omega^2) + j\omega c(\omega^2)}{a(\omega^2)} \right) \right| < \gamma, \tag{9.21}$$

parameterized by $\omega \in \mathbf{R}$. The inequalities defined by (9.20) are all linear. On the other hand, multiplying (9.21) by $a(\omega^2)$ yields an equivalent set of inequalities

$$\left| W(j\omega) \left(G(j\omega)a(\omega^2) - b(\omega^2) - j\omega c(\omega^2) \right) \right| < \gamma a(\omega^2).$$

Since

$$|z| = \max_{|w|=1} \mathrm{Re}(wz) \ \ \forall \ z \in \mathbf{C},$$

the last condition is equivalent to

$$\mathrm{Re} \left(w \left(G(j\omega)a(\omega^2) - b(\omega^2) - j\omega c(\omega^2) \right) \right) < \gamma a(\omega^2) \ \ \forall \ |w| \leq |W(j\omega)|, \tag{9.22}$$

which is a family of linear inequalities parameterized by w and ω. ∎

From this point on, the relaxed model reduction problem can be treated the same way the real rational approximation problem was treated, including replacing ω with $\omega_0 \tan(u/2)$, using root calculations within the feasibility oracle, finding an initial bounding ellipsoid, etc.

9.5.6 Theoretical properties of relaxed reduced model

One the minimum $\tilde{\sigma}_{m+1}(G)$ and an optimal (or a good quality suboptimal) solution $(q, p, r) \in \Omega^m_{qpr}$ of the relaxed mdel reduction problem are available, they can be used as "ad hoc" solutions in the original weighted H-Infinity model reduction problem: $\tilde{\sigma}_{m+1}$ becomes a lower bound for the minimum in the original problem, while $\hat{G}_{rlx} = p/q$ can be used as the reduced model.

The following theorem states that the relaxed convex optimization approach provides information about the original problem which is at least as valuable as the data produced by the balanced truncation algorithm.

Theorem 9.4 *Assume that $W \equiv 1$. Then*

(a) $\tilde{\sigma}_{m+1}(G) > \sigma_{m+1}(G)$ for all stable transfer functions G, of order larger than m;

(b) $\|G - \hat{G}_{rlx}\|_\infty \leq m\tilde{\sigma}_{m+1}(G)$.

The proof of Theorem 9.4 will be given in a future lecture, as it requires better understanding of Hankel singular numbers.

Item (a) of Theorem 9.4 states that the relaxation approach provides better lower bounds than those given by the Hankel singular numbers. Item (b)] shows that, in the relaxation-based model reduction, an a-priori upper bound for the H-Infinity error of model reduction can be given in terms of $\tilde{\sigma}_{m+1}(G)$ alone.

9.5.7 Sampled data model reduction

When the order of G is large, exact calculation of $\Phi(a, b, c)$ (L-Infinity approximation error in the relaxed setup) becomes prohibitively expensive. A significant benefit of the relaxation approach is that a lower bound $\Phi^-(a, b, c)$ of $\Phi(a, b, c)$ can be calculated with the knowledge of *frequency samples* of G:

$$\Phi^-(a, b, c) = \max_k \left| W(j\omega_k) \left(G(j\omega_k) - \frac{b(\omega_k^2) + j\omega_k c(\omega_k^2)}{a(\omega_k^2)} \right) \right|, \tag{9.23}$$

where $(\omega_k)_{k=1}^N$ is a selected set of frequencies. Quasi-convexity of Φ^- follows easily using the standard arguments. The minimum of Φ^- still provides a lower bound for the original weighted H-Infinity model reduction problem. When G is defined by a state space model of order n which is much larger than m, calculating the samples $G(j\omega_k)$ becomes the most expensive part of model reduction process.