Massachusetts Institute of Technology

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6.242, Fall 2004: MODEL REDUCTION [*]

Problem set 1[1]

Problem 1.1

For all values of parameter $a \in \mathbf{R}$, find the order of the LTI system with transfer matrix

$$H(s) = \frac{1}{s+1} \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}.$$

Optional: what is the relation between the order of $H(s) = M/(s+1)$ and the rank of matrix M?

Problem 1.2

LTI system with impulse response

$$g(t) = u(t) - u(t-1)$$

is approximated by the first order system with transfer function $\hat{G}(s) = 1/(1+0.5s)$. Find (approximately) the H-Infinity norm of the approximation error system.
Hint: calculate $G(s)$ analytically and use frequency sampling.

Problem 1.3

For all values of parameter $a \in \mathbf{R}$, find L2 gain of system

$$f(t) \mapsto y(t) = |f(t)| - f(t-a).$$

[1]Version of September 15, 2004. Due September 22, 2004.

Problem 1.4

A feedback design setup consists of a heat source supplying a controlled amount $f = f(t)$ of heat to one end of a homogeneous beam, and a sensor measuring the temperature $y = y(t)$ at the other end of the beam. The distribution $v = v(t, \theta)$ of temperature along the normalized length of the beam (from one end at $\theta = 0$ to the other end at $\theta = 1$) is described by the heat equation

$$\frac{dv(t,\theta)}{dt} = \frac{d^2 v(t,\theta)}{d\theta^2}$$

with boundary conditions

$$\left.\frac{dv(t,\theta)}{d\theta}\right|_{\theta=0} = -f(t), \quad \left.\frac{dv(t,\theta)}{d\theta}\right|_{\theta=1} = 0.$$

A proportional feedback

$$f(t) = K(r(t) - y(t)) = K(r(t) - v(t,1)),$$

where $r = r(t)$ is the reference input (the desired temperature at the $\theta = 1$ end of the beam) is proposed to control $y(t)$.

It is expected that using a larger value of the feedback gain K will result in a faster closed loop response. On the other hand, using a value of K which is too large will destabilize the feedback system. To predict the closed loop behavior, a reduced model of the true system is proposed, based on replacing the original PDE with an approximation $\hat{G}_n$ of order $n - 1$:

$$\begin{aligned}
\dot{v}_1 &= n^2(v_2 - v_1) + nf, \\
\dot{v}_k &= n^2(v_{k-1} + v_{k+1} - 2v_k), \quad (k = 2, \dots, n-2) \\
\dot{v}_{n-1} &= n^2(v_{n-2} - v_{n-1}), \\
y &= v_{n-1},
\end{aligned}$$

where $n > 3$ is an integer parameter. Here it is expected that

$$\begin{aligned}
v_k(t) &\approx v(t, k/n), \\
v_1(t) + f(t)/n &\approx v(t, 0), \\
v_{n-1}(t) &\approx v(t, 1).
\end{aligned}$$

(a) For all n, find matrices A, B, C, D of the state space model of the approximating system $\hat{G}_n$, assuming that its state is

$$x(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_{n-1}(t) \end{bmatrix}.$$

(b) For $n = 4, 10, 100$ find (approximately) the maximal $K_0 > 0$ such that $\hat{G}_n$ is stabilized by the feedback $f(t) = -Ky(t)$ for all $K \in (0, K_0)$.
Hint: you can use Bode plots of $\hat{G}_n$ generated by MATLAB.

(c) Find an analytical expression for the transfer function $G = G(s)$ of the original system.
Hint: look for a solution $v(t, \theta) = w(\theta, s)e^{st}$ of the system equations with $f(t) = e^{st}$.

(d) Find analytically the constant $\rho = \rho_n$ such that the difference $G - \rho_n \hat{G}_n$ has no unstable poles. Calculate (approximately) the H-Infinity norm of $G - \rho_n \hat{G}_n$ for $n = 4, 10, 100$.

(e) Use the small gain theorem and the results from (a),(b), and (d) to estimate the maximal K_0 such that G is stabilized by the feedback $f(t) = -Ky(t)$ for all $K \in (0, K_0)$.

(f) Use the Bode plot of G to check accuracy of the result from (e).