Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.242, Fall 2004: MODEL REDUCTION *

Problem set 1 solutions[1]

Problem 1.1

FOR ALL VALUES OF PARAMETER $a \in \mathbf{R}$, FIND THE ORDER OF THE LTI SYSTEM WITH TRANSFER MATRIX

$$H(s) = \frac{1}{s+1} \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}. \tag{1.1}$$

Optional: WHAT IS THE RELATION BETWEEN THE ORDER OF $H(s) = M/(s+1)$ AND THE RANK OF MATRIX M?

The order of system $H(s) = M/(s+1)$ equals the rank of M. In particular, for (1.1), the order is 2 when $a \neq 1$ and 1 when $a = 1$.

To prove the statement, let n be the rank of M. Then $M = FL$, where F, L are real matrices of rank n and of dimensions m-by-n and n-by-k respectively. A state space model of H with n states is given by

$$\dot{x} = -x + Lf, \quad y = Fx.$$

Since both controllability matrix

$$M_c = \begin{bmatrix} L & -L & L & \dots \end{bmatrix}$$

[1] Version of October 1, 2004.

and observability matrix

$$M_o = \begin{bmatrix} F \\ -F \\ F \\ \vdots \end{bmatrix}$$

have rank n, the state space model is minimal, and hence the order of H equals n.

Problem 1.2

LTI SYSTEM WITH IMPULSE RESPONSE

$$g(t) = u(t) - u(t-1)$$

IS APPROXIMATED BY THE FIRST ORDER SYSTEM WITH TRANSFER FUNCTION $\hat{G}(s) = 1/(1+0.5s)$. FIND (APPROXIMATELY) THE H-INFINITY NORM OF THE APPROXIMATION ERROR SYSTEM.

The transfer function of the original system is given by

$$G(s) = \frac{1-e^{-s}}{s}.$$

A simple-minded algorithm for numerical calculation of $\|G - \hat{G}\|_\infty$ can be based on evaluating $|G(j\omega) - \hat{G}(j\omega)|$ at

$$\omega = \Omega/N, 2\Omega/N, 3\Omega/N, \dots$$

To check the accuracy of the algorithm, note that

$$|dG(j\omega)/d\omega| \le 1, \quad |d\hat{G}(j\omega)/d\omega| \le 0.5 \;\; \forall\, \omega \in \mathbf{R},$$

and

$$|G(j\omega)| < 2/\Omega, \quad |\hat{G}(j\omega)| < 2/\Omega| \;\; \forall$$

Hence the error from sampling does not exceed

$$\frac{3}{2} \cdot \frac{\Omega}{2N} = \frac{3\Omega}{4N},$$

and the error from using a finite frequency range does not exceed

$$\frac{2}{\Omega} + \frac{2}{\Omega} = \frac{4}{\Omega}.$$

The total accuracy (in the case of precise arithmetic) would be

$$\frac{4}{\Omega} + \frac{3\Omega}{4N} \leq \frac{2\sqrt{3}}{\sqrt{N}},$$

which is maximized at $\Omega = 4\sqrt{N}/\sqrt{3}$.

The actual calculation is performed by the following MATLAB function.

```
function E=ps12_6242_2004(N)
% function E=ps12_6242_2004(N)
%
% estimates H-Infinity norm of (1-exp(-s))/s-1/(1+0.5s)
% larger N means better quality of estimation

if nargin<1, N=10000; end        % default number of samples
W=4*sqrt(N/3);                   % optimal W
e=8/W;                           % error bound
w=(1:N)*W/N;                     % w-samples
s=j*w;                           % s-samples
G=(1-exp(-s))./s;                % G-hamples
Ghat=1./(1+0.5*s);               % Ghat-samples
E=max(abs(G-Ghat));              % calculated H-Infinity norm
fprintf('\nThe norm is between %f and %f\n',E-e,E+e);
close(gcf)
subplot(2,1,1); plot(w,real(G),w,real(Ghat)); grid
subplot(2,1,2); plot(w,imag(G),w,imag(Ghat)); grid
```

The modeling error norm turns out to be about 0.3957.

Problem 1.3

For all values of parameter $a \in \mathbf{R}$, find L2 gain of system

$$f(t) \mapsto y(t) = |f(t)| - f(t-a).$$

The answer is 2 for $a \geq 0$ and ∞ for $a < 0$.

To show that the gain is not smaller than 2, consider the input $f_0(t) \equiv -1$, producing the output $y_0(t) \equiv 2$. Since every $\gamma \geq 0$ for which the integrals

$$\int_0^T \{\gamma^2 |f_0(t)|^2 - |y_0(t)|^2\} dt = T(\gamma^2 - 4)$$

are bounded from below as $T \to +\infty$ must satisfy $\gamma \geq 2$, we conclude that the L2 gain of the system is not smaller than 2.

To show that that gain is not larger than 2 for $a \geq 0$, note that

$$|f_1 + f_2|^2 \leq 2(|f_1|^2 + |f_2|^2)$$

for all real numbers f_1, f_2, and hence

$$\int_0^T |(|f(t)| - f(t-a))|^2 dt \leq 2 \int_0^T |f(t)|^2 dt + 2 \int_0^T |f(t-a)|^2 dt$$

$$\leq 2 \int_{-a}^0 |f(t)|^2 dt + 4 \int_0^T |f(t)|^2 dt.$$

Therefore

$$\int_0^T \{4|f_0(t)|^2 - |y_0(t)|^2\} dt$$

is bounded from below by the constant

$$2 \int_{-a}^0 |f(t)|^2 dt,$$

which does not depend on T. Hence L2 gain is not larger than 2 for $a \geq 0$.

Finally, to show that the gain is infinite for $a < 0$, consider the input

$$f_h(t) = e^{ht} u(t) = \begin{cases} e^{ht}, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

where $h > 0$ is a parameter. Then, for $t \geq 0$, the corresponding output $y = y_h(t)$ satisfies

$$|y_h| \geq e^{ht} |e^{ah} - 1|,$$

and hence

$$\int_0^T |y_h(t)|^2 dt = \frac{e^{2hT} - 1}{2h} |e^{ah} - 1|^2.$$

Since

$$\int_0^T |f_h(t)|^2 dt = \frac{e^{2hT} - 1}{2h},$$

the integral

$$\int_0^T \{\gamma^2 |f_h(t)|^2 - |y_h(t)|^2\} dt$$

converges to minus infinity for every $\gamma \geq 0$ when $h > 0$ is sufficiently large (dependent on γ).

Problem 1.4

A feedback design setup consists of a heat source supplying a controlled amount $f = f(t)$ of heat to one end of a homogeneous beam, and a sensor measuring the temperature $y = y(t)$ at the other end of the beam. The distribution $v = v(t, \theta)$ of temperature along the normalized length of the beam (from one end at $\theta = 0$ to the other end at $\theta = 1$) is described by the heat equation

$$\frac{dv(t,\theta)}{dt} = \frac{d^2 v(t,\theta)}{d\theta^2}$$

with boundary conditions

$$\left.\frac{dv(t,\theta)}{d\theta}\right|_{\theta=0} = -f(t), \quad \left.\frac{dv(t,\theta)}{d\theta}\right|_{\theta=1} = 0.$$

A proportional feedback

$$f(t) = K(r(t) - y(t)) = K(r(t) - v(t,1)),$$

where $r = r(t)$ is the reference input (the desired temperature at the $\theta = 1$ end of the beam) is proposed to control $y(t)$.

It is expected that using a larger value of the feedback gain K will result in a faster closed loop response. On the other hand, using a value of K which is too large will destabilize the feedback system. To predict the closed loop behavior, a reduced model of the true system is proposed, based on replacing the original PDE with an approximation $\hat{G}_n$ of order $n - 1$:

$$\begin{aligned}
\dot{v}_1 &= n^2(v_2 - v_1) + nf, \\
\dot{v}_k &= n^2(v_{k-1} + v_{k+1} - 2v_k), \quad (k = 2, \ldots, n-2) \\
\dot{v}_{n-1} &= n^2(v_{n-2} - v_{n-1}), \\
y &= v_{n-1},
\end{aligned}$$

WHERE $n > 3$ IS AN INTEGER PARAMETER. HERE IT IS EXPECTED THAT

$$\begin{aligned} v_k(t) &\approx v(t, k/n), \\ v_1(t) + f(t)/n &\approx v(t, 0), \\ v_{n-1}(t) &\approx v(t, 1). \end{aligned}$$

(a) FOR ALL n, FIND MATRICES A, B, C, D OF THE STATE SPACE MODEL OF THE APPROXIMATING SYSTEM $\hat{G}_n$, ASSUMING THAT ITS STATE IS

$$x(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_{n-1}(t) \end{bmatrix}.$$

We have

$$A = n^2 \begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 1 & -2 & 1 & & & \vdots \\ 0 & 1 & -2 & & & \\ & & & \ddots & & \\ \vdots & & & & -2 & 1 \\ 0 & \dots & & 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} n \\ 0 \\ \vdots \\ \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \dots & & 0 & 1 \end{bmatrix}, \quad D = 0.$$

Here is a MATLAB function generating the state space model:

```
function Gn=ps14a_6242_2004(n)
% function Gn=ps14a_6242_2004(n)
%
% solves Problem 1.4(a) from 6.242/2004
n2=n^2;
A=toeplitz([-2*n2;n2;zeros(n-3,1)]);
A(1,1)=-n2;
A(n-1,n-1)=-n2;
B=[n;zeros(n-2,1)];
C=[zeros(1,n-2) 1];
Gn=ss(A,B,C,0);
```

(b) FOR $n = 4, 10, 100$ FIND (APPROXIMATELY) THE MAXIMAL $K_0 > 0$ SUCH THAT $\hat{G}_n$ IS STABILIZED BY THE FEEDBACK $f(t) = -Ky(t)$ FOR ALL $K \in (0, K_0)$.

The control system toolbox function `margin.m` can do the job, but it gets confused when n reaches the 100 level. This software glitch can be fixed by working with re-scaled A, B. More precisely dividing A by n^2 and dividing B by n reduces the gain margin by a factor of n. The MATLAB code is shown below.

```
function g=ps14b_6242_2004(n)
% function g=ps14b_6242_2004(n)
%
% gain margin calculation for Problem 1.4(b) in 6.242/2004
[A,B,C]=ssdata(ps14a_6242_2004(n));     % get A,B,C
A=A/(n^2);                              % re-scale
B=B/n;
g=n*margin(ss(A,B,C,0));                % calculate the margin
```

The resulting gain margin is (approximately) 48.0 for $n = 4$, 20.9 for $n = 10$, and 17.9 for $n = 100$.

(c) FIND AN ANALYTICAL EXPRESSION FOR THE TRANSFER FUNCTION $G = G(s)$ OF THE ORIGINAL SYSTEM.

For a fixed $s > 0$, and for $f(t) = e^{st}$, the "steady state" response is to be given by $v(t, \theta) = u(\theta)e^{st}$, in which case $G(s) = u(1)$ is the desired transfer function from f to y. Substituting the expression for v into the PDE yields an parameterized ODE for $u = u(\theta)$:

$$\ddot{u}(\theta) = su(\theta), \ \dot{u}(0) = 1, \ \dot{u}(1) = 0.$$

Solving this ODE yields

$$u(\theta) = -\frac{1}{\sqrt{s}} \frac{e^{(\theta-1)\sqrt{s}} - e^{-(\theta-1)\sqrt{s}}}{e^{-\sqrt{s}} - e^{\sqrt{s}}},$$

hence

$$G(s) = \frac{2}{\sqrt{s}(e^{\sqrt{s}} - e^{-\sqrt{s}})}.$$

(d) FIND ANALYTICALLY THE CONSTANT $\rho = \rho_n$ SUCH THAT THE DIFFERENCE $G - \rho_n \hat{G}_n$ HAS NO UNSTABLE POLES. CALCULATE (APPROXIMATELY) THE H-INFINITY NORM OF $G - \rho_n \hat{G}_n$ FOR $n = 4, 10, 100$.

Matrix A is symmetric. Hence there exists an orthogonal basis in $\mathbf{R}^{n-1}$ consisting of eigenvectors v_k of A, i.e. $Av_k = \lambda_k v_k$, and $v_i' v_k = 0$ for $i \neq k$. One eigenvector of A is easy to guess from the physics of the setup: any constant temperature distribution is an equilibrium when $f \equiv 0$. This leads to

$$e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

as an eigenvector of A: $Ae = 0$. hence, one can think that $v_1 = e$ and $\lambda_1 = 0$. Since

$$x'Ax = -n^2 \sum_{k=1}^{n} -2(x_k - x_{k+1})^2$$

is non-negative only when $x = eq$ for some $q \in \mathbf{R}$, $\lambda_k < 0$ for $k = 2, \dots, n-1$.
Let

$$B = \sum_{i=1}^{n-1} v_i q_i$$

be the decomposition of B as a linear combination of the eigenvectors of A. Then

$$G_n(s) = \sum_{i=1}^{n-1} \frac{q_i C v_i}{s - \lambda_i},$$

and hence the unstable part $G_n^-(s)$ of $G_n(s)$ is given by

$$G_n^-(s) = \frac{q_1 C v_1}{s}.$$

Taking into account that

$$\lim_{s \to 0} sG(s) = 1,$$

the coefficient ρ_n must be equal to $1/q_1 C v_1$. Since $v_2, \dots, v_{n-1}$ are orthogonal to $v_1 = e$, it is sufficient to represent B as $B = q_1 e + B_\perp$, where $B_\perp$ is orthogonal to e.

Since

$$B_\perp = B - q_1 e = \begin{bmatrix} n - q_1 \\ -q_1 \\ \vdots \\ -q_1 \end{bmatrix},$$

this yields

$$n - q_1 - (n-2)q_1 = 0, \quad \text{i.e.} \quad q_1 = \frac{n}{n-1}.$$

Hence

$$\rho_n = \frac{n-1}{n}.$$

The following MATLAB code calculates the H-Infinity approximation error

```
function [g,rGns]=ps14d_6242_2004(n)
% function [g,rGns]=ps14d_6242_2004(n)
%
% H-Infinity error calculation for Problem 1.4(d) in 6.242/2004
N=10000;                                        % number of samples
rGn=((n-1)/n)*ps14a_6242_2004(n);               % reduced model
w=(1:N)'*100/N;                                  % frequency samples
sqrts=((1+j)/sqrt(2))*sqrt(w);                  % sqrt(s) samples
rGns=freqresp(rGn,w);                           % rho(n)*Gn samples
rGns=squeeze(rGns);
Gs=2./(sqrts.*(exp(sqrts)-exp(-sqrts)));        % G samples
g=max(abs(rGns-Gs));
close(gcf)                                      % a graphic sanity check
subplot(2,1,1); plot(w,real(rGns),w,real(Gs)); grid
subplot(2,1,2); plot(w,imag(rGns)+1./w,w,imag(Gs)+1./w); grid
```

The error is bounded by 0.09 for $n = 4$, 0.04 for $n = 10$, and 0.004 for $n = 100$.

(e) Use the small gain theorem and the results from (a),(b), and (d) to estimate the maximal K_0 such that G is stabilized by the feedback $f(t) = -Ky(t)$ for all $K \in (0, K_0)$.

G is stabilized by the feedback $f(t) = -Ky(t)$ for all $K \in (0, K_0)$ if and only if G is stabilized by the feedback $f(t) = -K_1 y(t)$ for some $K_1 \in (0, K_0)$, and $G(j\omega) \notin (-\infty, -K_0^{-1})$ for all $\omega \in \mathbf{R}$.

To check stability for some $K_1 \in (0, K_0)$, form the feedback interconnection of G_n and K_1 (transfer function $G_K = K_1/(1 + K_1\rho_n G_n)$), check its stability, and then check that the product of the H-Infinity norms $\|G_K\|_\infty$ and $\|G - \rho_n G_n\|_\infty$ is less than 1.

To find the largest $K_0 > 0$ such that $G(j\omega) \notin (-\infty, -K_0^{-1})$ for all $\omega \in \mathbf{R}$, find the smallest real y which is within the $\|G - \rho_n G_n\|_\infty$ distance from the Nyquist plot of G_n.

The following MATLAB code does the calculations.

```
function K0=ps14e_6242_2004(n)
% function K0=ps14e_6242_2004(n)
%
% gain margin estimation for Problem 1.4(e) in 6.242/2004
[g,rGns]=ps14d_6242_2004(n);
rGn=((n-1)/n)*ps14a_6242_2004(n);           % reduced model
K1=1;                                       % some feedback gain
GK1=K1/(1+K1*rGn);                          % closed loop
[z,p,k]=zpkdata(GK1);                       % closed loop poles
sm=max(real(p{1}));
fprintf('\nStability margin:  %f',sm);
sg=norm(GK1,Inf)*g;
fprintf('\nSmall gain margin: %f',sg);
if (sm<0)&(sg<1),                           % stability check
    fprintf('\nNominal stabilty established');
    rGns=rGns(abs(imag(rGns))<=g);
    y=min(real(rGns)-sqrt(g^2-imag(rGns).^2));
    K0=-1/y;
else
    fprintf('\nNominal stabilty not established');
end
```

Using $n = 100$ yields $K_0 = 16.5$ as the lower bound for the gain margin of G.

(f) Use the Bode plot of G to check accuracy of the result from (e).

The calculation checks for a sign change in the samples of the imaginary part of

$G(j\omega)$, and uses the minimum y of the real part of $G(j\omega)$ at those samples to define $K_0 = -1/y$.

```
function K0=ps14f_6242_2004(N)
% function K0=ps14f_6242_2004(N)
%
% gain margin estimation for Problem 1.4(f) in 6.242/2004
w=(1:N)'*50/N;                                 % frequency samples
sqrts=((1+j)/sqrt(2))*sqrt(w);                  % sqrt(s) samples
Gs=2./(sqrts.*(exp(sqrts)-exp(-sqrts)));    % G samples
ir=imag(Gs);
ir=ir(1:N-1).*ir(2:N);
y=min(real(Gs(ir<=0)));
K0=-1/y;
close(gcf)
subplot(2,1,1); plot(w,real(Gs),w,repmat(y,N,1)); grid
subplot(2,1,2); plot(w,imag(Gs)./max(0.1,abs(imag(Gs)))); grid
```

The resulting gain margin is 17.79.