Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.242, Fall 2004: MODEL REDUCTION *

Problem set 2[1]

Problem 2.1

The goal of this assignment is to test the degree of freedom available when deriving reduced models using a projection method. Consider the standard state space model with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

(transfer function $G(s) = 1/s^2$). A pair of projection matrices V and U of dimensions 2-by-1 and 1-by-2, respectively, satisfying the usual condition $UV = 1$, would produce a reduced model with transfer function $\hat{G}(s) = k/(s-a)$. Decribe analytically the set of all possible pairs (a, k).

Problem 2.2

The goal of this assignment is to extend the results of Lecture 4 notes on moments matching. Consider state space models

$$G := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right), \quad \hat{G} = \left(\begin{array}{c|c} UAV & UB \\ \hline CV & D \end{array} \right),$$

where $UV = I_r$. Let $s_0 \in \mathbf{C}$ be a complex number for which both matrices $s_0 I_n - A$ and $s_0 I_r - UAV$ are invertible. Assume that the columns of matrices $(s_0 I_n - A)^{-k-1} B$

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belong to the range of V for $k = 0, \dots, N_V$. In addition, assume that the rows of matrices $C(s_0 I_n - A)^{-k-1}$ can be represented as linear combinations of the rows of U for $k = 0, \dots, N_U$. Depending on the numbers N_U and N_V only, how many moments of $G(s)$ and $\hat{G}(s)$ are *guaranteed* to match at $s = s_0$ under these assumptions?

(a) Design a numerical experiment to supply you with data for making a hypotheses about the answer.

(b) Formulate the general answer and prove it formally.

Problem 2.3

The goal of this assignment is to apply the results of Lecture 4 on moments matching and stability preservation in projection based model reduction of a large mass/spring chain, modeled by the system of differential equations

$$\begin{aligned} M\ddot{x}_1(t) + B\dot{x}_1(t) + n^2 K(2x_1(t) - x_2(t)) &= f(t), \\ M\ddot{x}_k(t) + B\dot{x}_k(t) + n^2 K(2x_k(t) - x_{k+1}(t) - x_{k-1}(t)) &= 0 \quad (k = 2, 3, \dots, 2n), \\ M\ddot{x}_{2n+1}(t) + B\dot{x}_{2n+1}(t) + n^2 K(2x_{2n+1}(t) - x_{2n}(t)) &= 0, \\ y(t) &= x_{n+1}(t), \end{aligned}$$

where M, B, K are given positive constants, $n > 0$ is a large integer, $x_i(t)$ is the (one-dimensional) deflection of the i-th mass, and $n^2 K$ is the spring coefficient of the spring connecting the i-th and the $i+1$-st mass, as well as the 1st and the last masses to fixed positions.

(a) Find matrices A, B, C, D (depending on n, M, B, K) of a state space model of the system, using

$$x(t) = [x_0(t); x_1(t); \dots; x_{2n}(t); \dot{x}_0(t); \dot{x}_1(t); \dots; \dot{x}_{2n}(t)]$$

as the state vector.

(b) Find a matrix $P = P'$ such that, for the state space model from (a) with $f \equiv 0$, $x(t)' P x(t)$ is the sum of kinetic and potential energies of the system.

(c) For $M = 1$, $B = 0.2$, $K = 4$, and $n = 50$, compute numerically matrix V with 10 columns which form an orthonormal basis in the space of all linear combinations of vectors $A^{-k} B$ with $k = 1, 2 \dots, 10$.

(d) For V defined in (c), compute numerically $U = (V'PV)^{-1}V'P$, and form the corresponding projection reduced system $\hat{G}_1$.

(e) For V defined in (c), and for $U = V'$, compute numerically the corresponding projection reduced system $\hat{G}_2$.

(f) Use Bode plots to compare the quality of approximation of the original transfer function G by the reduced models $\hat{G}_1$ and $\hat{G}_2$.