Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.242, Fall 2004: MODEL REDUCTION *

Problem set 2 solutions[1]

Problem 2.1

The goal of this assignment is to test the degree of freedom available when deriving reduced models using a projection method. Consider the standard state space model with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

(transfer function $G(s) = 1/s^2$). A pair of projection matrices V and U of dimensions 2-by-1 and 1-by-2, respectively, satisfying the usual condition $UV = 1$, would produce a reduced model with transfer function $\hat{G}(s) = k/(s-a)$. Decribe analytically the set of all possible pairs (a, k).

The set consists of all pairs (a, k) such that $4ak \leq 1$.

To prove this, first note that $a = u_1 v_2$ and $k = u_2 v_1$, where the components of $U = [u_1 \;\; u_2]$ and $V = [v_1; v_2]$ must satisfy

$$u_1 v_1 + u_2 v_2 = 1.$$

Let $t = u_1 v_1$. Then

$$4ak = 4u_1 v_2 u_2 v_1 = 4u_1 v_1 u_2 v_2 = 4t(1-t) \leq 1.$$

[1]Version of October 4, 2004.

On the other hand, if $4ak \leq 1$ and $a \neq 0$ then $ak = t(1-t)$ for some $t \in \mathbf{R}$, hence a projection with

$$u_1 = 1,\ v_1 = t,\ v_2 = a,\ u_2 = (1-t)/a$$

generates the pair (a, k). Finally, when $a = 0$, a projection with

$$u_1 = 0,\ v_1 = k,\ u_2 = 1,\ v_2 = 1$$

generates the pair (a, k).

Problem 2.2

The goal of this assignment is to extend the results of Lecture 4 notes on moments matching. Consider state space models

$$G := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \quad \hat{G} = \left(\begin{array}{c|c} UAV & UB \\ \hline CV & D \end{array}\right),$$

where $UV = I_r$. Let $s_0 \in \mathbf{C}$ be a complex number for which both matrices $s_0 I_n - A$ and $s_0 I_r - UAV$ are invertible. Assume that the columns of matrices $(s_0 I_n - A)^{-k-1} B$ belong to the range of V for $k = 0, \dots, N_V$. In addition, assume that the rows of matrices $C(s_0 I_n - A)^{-k-1}$ can be represented as linear combinations of the rows of U for $k = 0, \dots, N_U$. Depending on the numbers N_U and N_V only, how many moments of $G(s)$ and $\hat{G}(s)$ are *guaranteed* to match at $s = s_0$ under these assumptions?

(a) Design a numerical experiment to supply you with data for making a hypotheses about the answer.

One appropriate numerical experiment can be described as follows. For a given pair of nonnegative integers N_U, N_V, generate a random number $m > \max\{N_U, N_V\}$, to become the order of the reduced system. Select randomly $n \gg 2 * m$ to become the order of the original system. Generate random n-by-n matrix A, column n-vector B, and a row n-vector C, as well as a complex number s_0. Form matrices U_0 and V_0 of dimensions m-by-n and n-by-m respectively, such that the first N_U rows of U_0 are $C(s_0 I - A)^{-i-1}$ for $i = 0, 1, \dots, N_U$, the first N_V columns of V_0 are $(s_0 I - A)^{-i-1} B$ for $i = 0, 1, \dots, N_V$, and the rest of rows/columns are generated randomly. Use singular value decomposition to produce better conditioned matrices U, V such that the row/columns spans of U and V are same as the row/column spans of U_0 and V_0. Re-define U according to

$$U := (UV)^{-1} U.$$

(Theoretically, matrix UV may turn out to be non-invertible, but, for moderately sized matrices, the probability of encountering such difficulty is small.)

The resulting MATLAB code `ps22a_6242_2004.m` is shown below.

```
function ps22a_6242_2004(nv,nu)
% function ps22a_6242_2004(nv,nu)
%
% reserach function for problem 2.2a

m=max(nu,nv)+ceil(3*rand);  % dimension of the reduced system
n=2*m+30+ceil(30*rand);     % randomized number of states
A=randn(n);                 % generate A,B,C
B=randn(n,1);
C=randn(1,n);
s0=randn+j*randn;          % generate s0
Ai=inv(s0*eye(n)-A);       % (s0I-A)^{-1}

U0=zeros(m,n);             % to store U0=[C(s0I-A)^{-1};C(s0I-A)^{-2};...]
Ck=C;
for i=1:nu+1,
    Ck=Ck*Ai;
    U0(i,:)=Ck;
end
U0(nu+2:m,:)=randn(m-nu-1,n);

V0=zeros(n,m);       % to store V0=[(s0I-A)^{-1}B;(s0I-A)^{-2}B;...]
Bk=B;
for i=1:nv+1,
    Bk=Ai*Bk;
    V0(:,i)=Bk;
end
V0(:,nv+2:m)=randn(n,m-nv-1);

[U,S]=svd(U0',0);
[V,S]=svd(V0,0);
U=inv(U'*V)*U';
A1=U*A*V;
B1=U*B;
```

```
C1=C*V;
n1=size(A1,1);

N=2*(nu+nv+2);
e=zeros(1,N);
A1i=inv(s0*eye(n1)-A1);
Bk=B;
B1k=B1;
for i=1:N,
    Bk=Ai*Bk;
    B1k=A1i*B1k;
    y(i)=C*Bk;
    y1(i)=C1*B1k;
end
close(gcf)
bar(abs(y-y1)./(1+abs(y)));grid
```

(b) Formulate the general answer and prove it formally.

Running the code from (a) suggests that **the first** $N_U + N_V + 2$ **moments are matched**. To prove this, let

$$B_i = (s_0 I_n - A)^{-i} B \ (i = 0, 1, \dots, N_V+1), \ \ C_i = C(s_0 I_n - A)^{-i} \ (i = 0, 1, \dots, N_U+1).$$

By construction,

$$B_i = V\hat{B}_i, \quad C_i = \hat{C}_i U \text{ for } i > 0.$$

As in the lecture notes, this implies

$$\hat{B}_i = (s_0 I_m - \hat{A})^{-i} \hat{B} \ (i = 0, 1, \dots, N_V+1), \ \ \hat{C}_i = \hat{C}(s_0 I_m - \hat{A})^{-i} \ (i = 0, 1, \dots, N_U+1).$$

Hence

$$C_i B_k = \hat{C}_i U V \hat{B}_k = \hat{C}_i \hat{B}_k$$

for

$$0 \le i \le N_U + 1, \ 0 \le k \le k \le N_V + 1, \ i + k > 0.$$

Since, for $i + k > 0$, $C_i B_k$ and $\hat{C}_i \hat{B}_k$ are the $(i+k)$-th moments of G and $\hat{G}$ at s_0, the proof is complete.

Problem 2.3

THE GOAL OF THIS ASSIGNMENT IS TO APPLY THE RESULTS OF LECTURE 4 ON MOMENTS MATCHING AND STABILITY PRESERVATION IN PROJECTION BASED MODEL REDUCTION OF A LARGE MASS/SPRING CHAIN, MODELED BY THE SYSTEM OF DIFFERENTIAL EQUATIONS

$$\begin{aligned} M\ddot{x}_1(t) + R\dot{x}_1(t) + n^2K(2x_1(t) - x_2(t)) &= f(t), \\ M\ddot{x}_k(t) + R\dot{x}_k(t) + n^2K(2x_k(t) - x_{k+1}(t) - x_{k-1}(t)) &= 0 \quad (k = 2, 3, \dots, 2n), \\ M\ddot{x}_{2n+1}(t) + R\dot{x}_{2n+1}(t) + n^2K(2x_{2n+1}(t) - x_{2n}(t)) &= 0, \\ y(t) &= x_{n+1}(t), \end{aligned}$$

WHERE M, R, K ARE GIVEN POSITIVE CONSTANTS, $n > 0$ IS A LARGE INTEGER, $x_i(t)$ IS THE (ONE-DIMENSIONAL) DEFLECTION OF THE i-TH MASS, AND n^2K IS THE SPRING COEFFICIENT OF THE SPRING CONNECTING THE i-TH AND THE $i+1$-ST MASS, AS WELL AS THE 1ST AND THE LAST MASSES TO FIXED POSITIONS.

(a) FIND MATRICES A, B, C, D (DEPENDING ON n, M, R, K) OF A STATE SPACE MODEL OF THE SYSTEM, USING

$$x(t) = [x_1(t); x_2(t); \dots; x_{2n+1}(t); \dot{x}_1(t); \dot{x}_1(t); \dots; \dot{x}_{2n+1}(t)]$$

AS THE STATE VECTOR.

The matrices have the form

$$A = \begin{bmatrix} 0 & I_{2n+1} \\ -\gamma & -(R/M)I_{2n+1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad C = \begin{bmatrix} c & 0 \end{bmatrix},$$

where

$$\gamma = \frac{n^2K}{M} \begin{bmatrix} 2 & -1 & 0 & & 0 \\ -1 & 2 & -1 & & 0 \\ 0 & -1 & & & \vdots \\ \vdots & & & \ddots & \\ 0 & 0 & \dots & & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1/M \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$c = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix}.$$

(b) FIND A MATRIX $P = P'$ SUCH THAT, FOR THE STATE SPACE MODEL FROM (A) WITH $f \equiv 0$, $x(t)'Px(t)$ IS THE SUM OF KINETIC AND POTENTIAL ENERGIES OF THE SYSTEM.

The kinetic energy of the i-th mass is $M\dot{x}_i^2/2$. The potential energy of the spring connecting the i-th and $(i+1)$-st mass is $K(x_i - x_{i+1})^2$. The first and the last springs have potential energies $n^2Kx_1^2/2$ and $n^2Kx_{2n+1}^2/2$ respectively. The total is defined by

$$P = \frac{M}{2}\begin{bmatrix} \gamma & 0 \\ 0 & I_{2n+1} \end{bmatrix}.$$

(c) FOR $M = 1$, $R = 0.2$, $K = 4$, AND $n = 50$, COMPUTE NUMERICALLY MATRIX V WITH 10 COLUMNS WHICH FORM AN ORTHONORMAL BASIS IN THE SPACE OF ALL LINEAR COMBINATIONS OF VECTORS $A^{-k}B$ WITH $k = 1, 2 \dots, 10$.

(d) FOR V DEFINED IN (C), COMPUTE NUMERICALLY $U = (V'PV)^{-1}V'P$, AND FORM THE CORRESPONDING PROJECTION REDUCED SYSTEM $\hat{G}_1$.

(e) FOR V DEFINED IN (C), AND FOR $U = V'$, COMPUTE NUMERICALLY THE CORRESPONDING PROJECTION REDUCED SYSTEM $\hat{G}_2$.

(f) USE BODE PLOTS TO COMPARE THE QUALITY OF APPROXIMATION OF THE ORIGINAL TRANSFER FUNCTION G BY THE REDUCED MODELS $\hat{G}_1$ AND $\hat{G}_2$.

The items (c)-(f) are done in MATLAB file `ps23cdef_6242_2004.m`, shown below. The resulting approximations $\hat{G}_1$ and $\hat{G}_2$ turn out to be quite good in the low frequency range, and get better as the number of moments matched increases.

```
function ps23cdef_6242_2004(n,m,M,R,K)
% function ps23cdef_6242_2004(n,m,M,R,K)
%
% Solves Problem 2.3, items (c)-(f)

if nargin<1, n=50; end
if nargin<2, m=10; end
if nargin<3, M=1; end
if nargin<4, R=0.2; end
```

```
if nargin<5, K=4; end

g=(2*(n^2)*K/M)*toeplitz([ 2 -1 zeros(1,2*n-1)]);
In=eye(2*n+1);
On=zeros(2*n+1);
b=[1;zeros(2*n,1)];
c=[zeros(1,n) 1 zeros(1,n)];
A=[On In; -g -(R/M)*In];
B=[zeros(2*n+1,1/M);b];
C=[c zeros(1,2*n+1)];
P=(M/2)*[g On;On In];

V=zeros(4*n+2,m);
Ai=inv(A);
Bk=Ai*B;
for i=1:m,
    Bk=Bk/norm(Bk);
    V(:,i)=Bk;
    Bk=Ai*Bk;
    Bk=Bk-V(:,1:i)*(V(:,1:i)'*Bk);
end

U=inv(V'*P*V)*V'*P;
A1=U*A*V;
B1=U*B;
C1=C*V;

A2=V'*A*V;
B2=V'*B;
C2=C*V;

w=linspace(0,100,1000);
g=squeeze(freqresp(ss(A,B,C,0),w));
g1=squeeze(freqresp(ss(A1,B1,C1,0),w));
g2=squeeze(freqresp(ss(A2,B2,C2,0),w));

close(gcf)
subplot(2,1,1);plot(w,real(g),w,real(g1)); grid
```

```
subplot(2,1,2);plot(w,real(g),w,real(g2)); grid
```