Massachusetts Institute of Technology

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6.242, Fall 2004: MODEL REDUCTION [*]

Problem set 3 solution[1]

Problem 3.1

Hankel singular numbers of a stable causal LTI system G of very large order are given by

$$\sigma_k(G) = 2^{-m} \quad \text{for} \quad k = 2^m + 1, \dots, 2^{m+1}, \quad (m = 0, 1, \dots), \quad \sigma_1(G) = 2.$$

(a) Suggest a good a-priori lower bound for the quality $\|G - \hat{G}\|_\infty$ of approximating G by a system $\hat{G}$ of order 8.

The available lower bound is $\sigma_9(G) = 1/8$.

(b) Suggest a good a-priori upper bound for the quality $\|G - \hat{G}_{btr}\|_\infty$ of approximating G by a system $\hat{G}_{btr}$ of order 8 using the method of balanced truncation.

The available upper bound equals the double sum of *different* singular values $\sigma_k(G)$ with $k > 8$, which yields

$$2(2^{-3} + 2^{-4} + 2^{-5} + \dots) = 2 \cdot 2^{-2} = 1/2.$$

[1]Version of October 26, 2004.

(c) What are the Hankel singular numbers of $\hat{G}_{btr}$ from (b)?

As it is proven in the lecture notes, Hankel sinular numbers of the 8-th order reduced system are the first eight Hankel numbers of G:

$$2,\ 1,\ 1/2,\ 1/2,\ 1/4,\ 1/4,\ 1/4,\ 1/4.$$

Problem 3.2

Give an explicit description of the set of all possible n-vectors

$$[\sigma_1(G), \sigma_2(G), \dots, \sigma_n(G)],$$

formed by the Hankel singular values $\sigma_k(G)$ of all stable "all pass" transfer functions G of order n (i.e. such that $|G(j\omega)| = 1$ for all ω).

(a) Design a numerical experiment, utilizing MATLAB functions `lyap`, `chol`, and `eig`, to collect data on the topic. (Do not use `sysbal` and similar "full service" model reduction functions.)

Function `hsvd_6242.m` provides calculation of Hankel singular values for stable systems of moderate order. (It can also produce a reduced order system, when necessary.)

```
function H=hsvd_6242(G,m)
% function H=hsvd_6242(G,m)
%
% Hankel svd for CT stable system G
%   with one argument:  H is the ordered vector of Hankel singular values
%   with two arguments: H is the m-th order btr reduced system

if nargin<1,   % a test example
    G=ss(diag(-(1:30)),ones(30,1),ones(1,30),0);
    m=3;
end

[A,B,C,d]=ssdata(G);
Wc=lyap(A,B*B');
Wo=lyap(A',C'*C);
[V,D]=eig(Wc*Wo);
```

```
V=real(V);
[W,I]=sort(-sqrt(abs(diag(D))));
V=V(:,I);
W=-W;
if nargin==1,
    H=W;
else
    V=V(:,1:m);
    U=(V'*Wo*V)\(V'*Wo);
    H=ss(U*A*V,U*B,C*V,d);
end

if nargin~=1,
    w=linspace(0,100,1000);
    g=squeeze(freqresp(G,w));
    g1=squeeze(freqresp(H,w));
    close(gcf)
    subplot(2,1,1);plot(w,real(g),w,real(g1)); grid
    subplot(2,1,2);plot(w,imag(g),w,imag(g1)); grid
end
```

Function `ps32_6242_2004.m` generates random all-pass systems and calculates their Hankel singular numbers, utilizing `hsvd_6242.m`.

```
function ps32_6242_2004(n,m)
% function ps32_6242_2004(n,m)
%
% solves Problem 3.2a by generating a random all-pass
% stable transfer function G of order n+2m (n real poles, 2n complex poles)
% and finding its Hankel singular values
%
% uses hsvd_6242.m

if nargin<1, n=5; end
if nargin<2, m=5; end

p=rand(n);    %  -p are real poles
a=rand(m);    %  -a are real parts of complex poles
```

```
b=rand(m);    %  b  are imaginary parts of complex poles

s=tf('s');
G=1;
for k=1:n,
    G=G*((s-p(k))/(s+p(k)));
end
for k=1:m,
    G=G*((s^2-2*a(k)*s+a(k)^2+b(k)^2)/(s^2+2*a(k)*s+a(k)^2+b(k)^2));
end

hsvd_6242(G)
```

(b) Formulate a hypotheses on what the answer is.

The numerical experiment indicates clearly that all Hankel singular numbers of an all-pass system equal 1.

(c) Prove the hypotheses (at least for the case $n = 2$).

Let

$$G(s) = C(sI - A)^{-1}B + D$$

be an all-pass stable transfer function. Without loss of generality, assume that A is an n-by-n Hurwitz matrix, the pair (A, B) is controllable, and the pair (C, A) is observable.

The most important step of the proof is to establish existence of a matrix $P = P'$ such that

$$2\bar{x}'P(A\bar{x} + B\bar{f}) = |\bar{f}|^2 - |C\bar{x} + D\bar{f}|^2 \;\; \forall\; \bar{f} \in \mathbf{R},\; \bar{x} \in \mathbf{R}^n, \tag{3.1}$$

which is a special case of the KYP (Kalman-Yakubovich-Popov) Lemma.

To prove (3.1) independently, define P as the observability Gramian $P = W_o$. By assumption,

$$|G(j\omega)|^2 - 1 = 0 \;\; \forall\; \omega \in \mathbf{R}.$$

Hence

$$\int_{-\infty}^{\infty}(|\tilde{y}(j\omega)|^2-|\tilde{f}(j\omega)|^2)d\omega=0,$$

where $\tilde{f}$ is the Fourier transform of a square integrable function $f=f(t)$ defined for $t\geq 0$, and $\tilde{y}(j\omega)=G(j\omega)\tilde{f}(j\omega)$. According to the Parceval identity, this means that

$$\int_0^{\infty}(|Cx(t)+Df(t)|^2-|f(t)|^2)dt=0$$

for the solution $x=x(t)$ of

$$\dot{x}(t)=Ax(t)+Bf(t),\quad x(0)=0. \tag{3.2}$$

Since

$$\int_T^{\infty}(|Cx(t)+Df(t)|^2-|f(t)|^2)dt=x(T)'W_ox(T)$$

whenever $f(t)\equiv 0$ for $t\geq T$, we have

$$x(T)'W_ox(T)+\int_0^{T}(|Cx(t)+Df(t)|^2-|f(t)|^2)dt\equiv 0$$

for *every* solution of (3.2). Differentiating this with respect to T yields

$$2x(T)'W_o(Ax(T)+f(T))+|Cx(T)+Df(T)|^2-|f(T)|^2\equiv 0$$

Since the pair (A,B) is controllable, $x(T)$ can be an arbitrary vector from $\mathbf{R}^n$. Since $f(T)$ can also be chosen arbitrarily (and independently of $x(T)$), identity (3.1) holds for $P=W_o$.

Once (3.1) is established, comparing the coefficients at fx on both sides of the identity yields

$$B'P+DC=0.$$

In addition, comparing the coefficients at f^2 yields $D^2=1$. Substituting $C=-D^{-1}B'W_o$ into the Lyapunov equation

$$W_oA+A'W_o=-C'C$$

and multiplying by W_o^{-1} on both sides yields

$$AW_o^{-1}+W_o^{-1}A'=-BB'.$$

Hence $W_c=W_o^{-1}$, and the Hankel singular values of G, as square roots of the eigenvalues of $W_oW_c=I_n$, are all equal to 1.

Problem 3.3

Use the method of balanced truncation to find a 10th order reduced model for the system described in Problem 2.3, with $M = 1$, $B = 0.2$, $K = 4$, and $n = 50$. (Do not use `sysbal` and similar "full service" model reduction functions.)

The task is performed by `ps33_6242_2004.m`. Applying balanced truncation in the case $M = 1$, $B = 0.2$, $K = 4$, $n = 50$ fails miserably, which can explained by the fact that there are no 10th order approximations which are good "accross the spectrum" in this case. On the other hand, for a larger dissipation factor $B = 20$, balanced truncation performs much better than the moments matching methods from problem set 2.

```
function ps33_6242_2004(n,m,M,R,K)
% function ps33_6242_2004(n,m,M,R,K)
%
% Solves Problem 3.3

if nargin<1, n=50; end
if nargin<2, m=10; end
if nargin<3, M=1; end
if nargin<4, R=0.2; end
if nargin<5, K=4; end

g=(2*(n^2)*K/M)*toeplitz([ 2 -1 zeros(1,2*n-1)]);
In=eye(2*n+1);
On=zeros(2*n+1);
b=[1;zeros(2*n,1)];
c=[zeros(1,n) 1 zeros(1,n)];
A=[On In; -g -(R/M)*In];
B=[zeros(2*n+1,1);b];
C=[c zeros(1,2*n+1)];

G=ss(A,B,C,0);
H=hsvd_6242(G,m);
```