Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.242, Fall 2004: MODEL REDUCTION *

Problem set 4[1]

Problem 4.1

The objective of this assignment is to learn how to use basic properties of Lyapunov equations to prove certain properties of controllability and observability Gramians of continuous time systems.

(a) Find an explicit formula which, given n different positive numbers $\sigma_1 > \sigma_2 > \cdots > \sigma_{n-1} > \sigma_n$, produces matrices A, B, C of a controllable and observable SISO stable state space model for which the k-th Hankel singular number equals σ_k. **Hint:** this can be accomplished by using a *symmetric* matrix A.

(b) Verify your formula using numerical calculations with MATLAB.

(c) Prove or give a counterexample to the following statement: *a SISO system defined by a controllable and observable state space model with $A = A' < 0$ and $B = C'$ cannot have repeated non-zero Hankel singular values.* **Hint:** check controllability of (W, B).

(d) **Optional:** find an explicit formula which, given n different positive numbers $\sigma_1 > \sigma_2 > \cdots > \sigma_{n-1} > \sigma_n$, and n different positive integers $m_1 < m_2 < \cdots < m_n$, produces matrices A, B, C of a controllable and observable SISO stable state space model for which the i-th Hankel singular number equals σ_k for $m_{k-1} < i \leq m_k$, $k = 1, \ldots, n$, where $m_0 = 0$.

[1]Version of October 13, 2004. Due October 20, 2004.

Problem 4.2

The objective of this assignment is to explore application of the POD method to infinite dimensional causal SISO CT LTI systems with explicitly known impulse response $h = h(t)$, assumed to be a continuous function $h : \mathbf{R} \mapsto \mathbf{R}$ (for example, the impulse respose corresponding to $G(s) = e^{-s}/(s+1)^2$ is continuous, while the impulse response corresponding to $G(s) = e^{-s}/(s+1)$ is not).

An abstract state space model can be defined for such systems in the following way.

- The "state vector" $x(t)$ is, for every $t \in \mathbf{R}$, a continuous function $x_t : \mathbf{R} \mapsto \mathbf{R}$, i.e. $x(t) = x(t, \tau)$.
- The linear transformation A is introduced indirectly, via its exponent: $\bar{x}_t = e^{At}\bar{x}_0$, where $\bar{x}_t, \bar{x}_0$ are vectors in the state space (i.e. functions of $\tau \in \mathbf{R}$) is defined by

  $$\bar{x}_t(\tau) = \bar{x}_0(t + \tau)$$

  for all t, τ.
- B (also a vector from the state space, and, hence, a function of τ) is defined by $B(\tau) = h(\tau)$.
- C (a linear functional on the state space) is defined by $C\bar{x} = \bar{x}(0)$.

Note that $h(t) = Ce^{At}B$ for this model.

To apply the POD method to reduce this model, select a finite set of linear functionals $L_1, L_2, \dots, L_N$ on the state space, as well as vectors $F_1, F_2, \dots, F_N$ from the state space. Treat the N-dimensional vectors

$$\tilde{x}(t) = \begin{bmatrix} L_1 e^{At} B \\ L_2 e^{At} B \\ \vdots \\ L_N e^{At} B \end{bmatrix}, \quad \tilde{p}(t) = \begin{bmatrix} Ce^{At} F_1 \\ Ce^{At} F_2 \\ \vdots \\ Ce^{At} F_N \end{bmatrix}$$

as the primal ad dual "snapshots". Calculate explicitly the integrals

$$\tilde{W}_c = \int_0^\infty \tilde{x}(t)\tilde{x}(t)' dt, \quad \tilde{W}_o = \int_0^\infty \tilde{p}(t)\tilde{p}(t)' dt,$$

to serve as "approximations" of the controllability and observability Gramians. Use singular value decomposition to produce vectors $v_1, v_2, \dots, v_r$ (linear combinations of vectors

F_i), and functionals $u_1, u_2, \ldots, u_r$ (linear combinations of functionals L_i), so that $r \ll N$, and the reduced model should be defined by

$$\hat{C} = \left[\begin{array}{cccc} Cv_1 & Cv_2 & \ldots & Cv_k \end{array}\right], \quad \hat{B} = \left[\begin{array}{c} u_1 B \\ u_2 B \\ \vdots \\ u_k B \end{array}\right], \quad \hat{D} = 0,$$

$$\hat{A}_{ij} = u_i A v_j = \lim_{t \to 0} \frac{1}{t} u_i (e^{At} v_j - v_j).$$

Implement the approach described above in the case when

$$h(t) = \begin{cases} t - 1, & t \in [1, 2], \\ 3 - t, & t \in [2, 3], \\ 0, & \text{otherwise}, \end{cases}$$

$$F_k(\tau) = h(\tau + 4k/N), \quad L_k \bar{x} = \bar{x}(4k/N).$$

(a) Find analytical expressions for $\tilde{W}_c, \tilde{W}_o$.

(b) Propose an algorithm (relaying on numerical singular value decomposition) for constructing u_i, v_i when $r \ll N$.

(c) Implement the resulting model reduction algorithm in MATLAB and test the reduced models for $r \in \{5, 10\}$ and $N \in \{50, 500\}$.

Problem 4.3

The objective of this assignment is to analyze existence and uniqueness of solution in a particular class of moments matching problems.

Consider the following moments matching problem: given a sequence of $2n$ real numbers $(f_k)_{k=0}^{2n-1}$, find real polynomials p, q such that $\deg(q) = n$, $\deg(p) \le n - 1$, $q(0) \neq 0$, and f_k are the first $2n$ moments of $p(s)/q(s)$ at $s = 0$, i.e.

$$\frac{p(s)}{q(s)} = O(s^{2n}) + \sum_{k=0}^{2n-1} f_k s^k$$

for $s \to 0$.

(a) Show that there exists a unique non-negative $r \le n$ and a pair of real polynomials

$$\bar{p}(s) = \bar{p}_0 + \bar{p}_1 s + \cdots + \bar{p}_{r-1} s^{r-1},$$

$$\bar{q}(s) = 1 + \bar{q}_1 s + \cdots + \bar{q}_{r-1} s^{r-1} + \bar{q}_r s^r$$

with no common zeros, such that the first $n + r$ moments of $\bar{f}(s) = \bar{p}(s)/\bar{q}(s)$ at $s = 0$ are $f_0, f_1, \dots, f_{2n-r-1}$. **Hint:** define

$$f(s) = \sum_{k=0}^{2n-1} f_k s^k,$$

and consider the linear transformation mapping the coefficients of polynomials $\hat{p}, \hat{q}$ of order not exceeding $n - 1$ and n respectively, into the first $2n$ moments of $\hat{p} - \hat{q} f$. Use the fact that for every m-by-$(m+1)$ matrix M there exists a non-zero vector x such that $Mx = 0$.

(b) Let d be the maximal number of first moments of $\bar{f}(s)$ at $s = 0$ which are matching the numbers $f_0, f_1, \dots, f_{2n-1}$. Let n_p, n_q be the degrees of $\bar{p}$ and $\bar{q}$ respectively. In terms of numbers r, d, n_p, n_q, give necessary and sufficient conditions for existence and (separately) uniqueness of solutions of the original moments matching problem.

(c) Use the result from (b) to generate real sequences $(f_k)_{k=0}^{2n-1}$ with $n = 3$ and $f_k \neq 0$, such that the moments matching problem has no solution and (separately) has many solutions.