Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.242, Fall 2004: MODEL REDUCTION *

Problem set 4 solutions[1]

Problem 4.1

The objective of this assignment is to learn how to use basic properties of Lyapunov equations to prove certain properties of controllability and observability Gramians of continuous time systems.

(a) Find an explicit formula which, given n different positive numbers $\sigma_1 > \sigma_2 > \cdots > \sigma_{n-1} > \sigma_n$, produces matrices A, B, C of a controllable and observable SISO stable state space model for which the k-th Hankel singular number equals σ_k.

Balanced realizations offer a convenient way of constructing systems with prescribed Hankel singular numbers. Define

$$W = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}, \quad B = C' = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

and then let $A = A'$ be the solution of the Lyapunov equation

$$WA + AW = -BB'.$$

[1]Version of December 2, 2004.

Since $-W$ is a Hurwitz matrix, and the pair $(-W, B)$ is controllable, a solution $A = A' < 0$ exists, and defines a stable LTI system with transfer function $C(sI - A)^{-1}B$, which has σ_k as its Hankel singular values. In fact, an explicit formula for A can be extracted from this construction:

$$A = -\begin{bmatrix} \frac{1}{2\sigma_1} & \frac{1}{\sigma_1+\sigma_2} & \cdots & \frac{1}{\sigma_1+\sigma_n} \\ \frac{1}{\sigma_1+\sigma_2} & \frac{1}{2\sigma_2} & & \\ \vdots & & \ddots & \\ \frac{1}{\sigma_1+\sigma_n} & & & \frac{1}{2\sigma_n} \end{bmatrix}.$$

(b) VERIFY YOUR FORMULA USING NUMERICAL CALCULATIONS WITH MATLAB.

See MATLAB functions `ps41a_6242_2004.m` and `ps41b_6242_2004.m`. It appears that the method produces systems with extremely poor stability margins, which causes Hankel singular value calculations to become unreliable for $n \geq 10$.

(c) PROVE OR GIVE A COUNTEREXAMPLE TO THE FOLLOWING STATEMENT: *a SISO system defined by a controllable and observable state space model with $A = A' < 0$ and $B = C'$ cannot have repeated non-zero Hankel singular values.*

If $A = A'$ and $B = C'$ then controllability and observability Gramians W_c, W_o satisfy the same Lyaponov equation $WA + AW = -BB'$. Since A is negative definite, solution of the equation with respect to W is unique, and hence $W_c = W_o = W$ has repeated eigenvalues. hence the pair $(-W, B)$ cannot be controllable, and therefore $-A$ is not positive definite. This contradicts the assumption.

Problem 4.2

THE OBJECTIVE OF THIS ASSIGNMENT IS TO EXPLORE APPLICATION OF THE POD METHOD TO INFINITE DIMENSIONAL CAUSAL SISO CT LTI SYSTEMS WITH EXPLICITLY KNOWN IMPULSE RESPONSE $h = h(t)$, ASSUMED TO BE A CONTINUOUS FUNCTION $h: \mathbf{R} \mapsto \mathbf{R}$ (FOR EXAMPLE, THE IMPULSE RESPOSE CORRESPONDING TO $G(s) = e^{-s}/(s+1)^2$ IS CONTINUOUS, WHILE THE IMPULSE RESPONSE CORRESPONDING TO $G(s) = e^{-s}/(s+1)$ IS NOT).

AN ABSTRACT STATE SPACE MODEL CAN BE DEFINED FOR SUCH SYSTEMS IN THE FOLLOWING WAY.

- THE "STATE VECTOR" $x(t)$ IS, FOR EVERY $t \in \mathbf{R}$, A CONTINUOUS FUNCTION $x_t: \mathbf{R} \mapsto \mathbf{R}$, I.E. $x(t) = x(t, \tau)$.

- The linear transformation A is introduced indirectly, via its exponent: $\bar{x}_t = e^{At}\bar{x}_0$, where $\bar{x}_t, \bar{x}_0$ are vectors in the state space (i.e. functions of $\tau \in \mathbf{R}$) is defined by

$$\bar{x}_t(\tau) = \bar{x}_0(t+\tau)$$

for all t, τ.

- B (also a vector from the state space, and, hence, a function of τ) is defined by $B(\tau) = h(\tau)$.

- C (a linear functional on the state space) is defined by $C\bar{x} = \bar{x}(0)$.

Note that $h(t) = Ce^{At}B$ for this model.

To apply the POD method to reduce this model, select a finite set of linear functionals $L_1, L_2, \dots, L_N$ on the state space, as well as vectors $F_1, F_2, \dots, F_N$ from the state space. Treat the N-dimensional vectors

$$\tilde{x}(t) = \begin{bmatrix} L_1 e^{At}B \\ L_2 e^{At}B \\ \vdots \\ L_N e^{At}B \end{bmatrix}, \quad \tilde{p}(t) = \begin{bmatrix} Ce^{At}F_1 \\ Ce^{At}F_2 \\ \vdots \\ Ce^{At}F_N \end{bmatrix}$$

as the primal ad dual "snapshots". Calculate explicitly the integrals

$$\tilde{W}_c = \int_0^\infty \tilde{x}(t)\tilde{x}(t)'dt, \quad \tilde{W}_o = \int_0^\infty \tilde{p}(t)\tilde{p}(t)'dt,$$

to serve as "approximations" of the controllability and observability Gramians. Use singular value decomposition to produce vectors $v_1, v_2, \dots, v_r$ (linear combinations of vectors F_i), and functionals $u_1, u_2, \dots, u_r$ (linear combinations of functionals L_i), so that $r \ll N$, and the reduced model should be defined by

$$\hat{C} = \begin{bmatrix} Cv_1 & Cv_2 & \dots & Cv_k \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} u_1 B \\ u_2 B \\ \vdots \\ u_k B \end{bmatrix}, \quad \hat{D} = 0,$$

$$\hat{A}_{ij} = u_i A v_j = \lim_{t\to 0} \frac{1}{t} u_i(e^{At}v_j - v_j).$$

Implement the approach described above in the case when

$$h(t) = \begin{cases} t-1, & t \in [1,2], \\ 3-t, & t \in [2,3], \\ 0, & \text{otherwise}, \end{cases}$$

$$F_k(\tau) = h(\tau + 4k/N), \quad L_k \bar{x} = \bar{x}(4k/N).$$

(a) Find analytical expressions for $\tilde{W}_c, \tilde{W}_o$.

To simplify the formulae, it will be assumed that $N = 4n$ throughout the solution. Since N is simply the number of time domain samples used, this is not a serious limitation. For the same purpose, elements F_0, L_0 will be added to the families $\{F_i\}$, $\{L_i\}$, defined by

$$F_0(\tau) = h(\tau), \quad L_0(x(\cdot)) = x(0) = Cx(\cdot).$$

Let

$$e_i(t) = u(t) \max\{0, 1 - n|t - i/n|\},$$

where

$$u(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

is the unit step function. According to this definition $e_i = e_i(t)$ for $i > 0$ is a piecewise-linear continuous function with a triangular-shaped graph, equals 1 at $t = i/n$, and equals 0 outside the interval $[(i-1)/n, (i+1)/n]$. Let

$$F(t) = u(t)[h(t),\ h(t+1/n),\ \dots,\ h(t+4n/n)],$$

$$E(t) = [e_0(t),\ e_1(t),\ \dots,\ e_{4n}(t)].$$

It is easy to see that

$$F(t) = E(t)H,$$

where H is the Hankel matrix

$$H = H' = \begin{bmatrix} h_0 & h_1 & \dots & h_{4n-1} & h_{4n} \\ h_1 & h_2 & \dots & h_{4n} & 0 \\ \vdots & \vdots & \ddots & & \\ h_{4n-1} & h_{4n} & & \ddots & \\ h_{4n} & 0 & & & 0 \end{bmatrix}.$$

In addition, a straightforward integration shows that

$$\int_0^\infty E(t)'E(t)dt = \frac{1}{n}W,$$

where W is a tridiagonal matrix

$$W = \frac{1}{6}\begin{bmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 4 & 1 & \dots & 0 \\ 0 & 1 & 4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 4 \end{bmatrix}.$$

Since, for a function $x = x(\tau)$ (an element of the state space),

$$Ce^{At}x = x(t),$$

it follows that

$$\langle x, W_o x \stackrel{\text{def}}{=} \int_0^\infty |Ce^{At}x|^2 dt = \int_0^\infty |x(t)|^2 dt.$$

Hence, for

$$x(t) = \sum_{i=0}^{4n} g_i F_i(t),$$

the observability measure equals

$$\langle x, W_o x = \int_0^\infty |F(t)g|^2 dt = \int_0^\infty |E(t)Hg|^2 dt = \frac{1}{n} g' HWHg,$$

where g is the $(4n+1)$-dimensional column vector with coefficients g_i. Therefore

$$\tilde{W}_o = \frac{1}{n} HWH.$$

Similarly, for a linear functional

$$p = \sum_{i=0}^{4n} q_i L_i, \; px = \sum_{i=0}^{4n} q_i x(i/n),$$

the controllability integral equals

$$\begin{aligned}\langle p, W_c p\rangle &\overset{\text{def}}{=} \int_0^\infty \left|\sum_i q_i h(t+i/n)\right|^2 dt,\\ &= \int_0^\infty |F(t)q'|^2 dt,\\ &= \frac{1}{n} qHWHq',\end{aligned}$$

where q is the $(4n+1)$-dimensional row vector with elements q_i. Hence

$$\tilde{W}_c = \tilde{W}_o = \frac{1}{n} HWH.$$

(b) PROPOSE AN ALGORITHM (RELAYING ON NUMERICAL SINGULAR VALUE DECOMPOSITION) FOR CONSTRUCTING u_i, v_i WHEN $r \ll N$.

Remember that a joint measure of controllability and observability of a vector x in the state space can be defined as

$$\phi_p(x) = x' W_o x \cdot \inf_{p:\ px=1} p W_c p', \tag{4.1}$$

where W_c, W_o are the true controllability/observability Gramians, and p ranges over the space of linear functionals on the state space. A similar (dual) measure for functionals is defined by

$$\phi_d(p) = p W_c p' \cdot \inf_{x:\ px=1} x' W_o x. \tag{4.2}$$

The partial information about W_o, W_c available from $\tilde{W}_o$ and $\tilde{W}_c$ can be utilized in modified definitions of ϕ_p, ϕ_d, obtained by limiting the range of x, p in (4.1),(4.2) to

$$x = \sum_{i=0}^{4n} F_i g_i, \quad p = \sum_{i=0}^{4n} q_i L_i, \tag{4.3}$$

which leads to

$$\hat{\phi}_p(g) = g' \tilde{W}_o g \cdot \inf_{q:\ qMg=1} q \tilde{W}_c q', \tag{4.4}$$

$$\hat{\phi}_d(q) = q \tilde{W}_c q' \cdot \inf_{g:\ qMg=1} g' \tilde{W}_o g, \tag{4.5}$$

where $M = (M_{ij})$ is the $(4n+1)$-by-$(4n+1)$ matrix with entries

$$M_{ij} = L_i F_j = h((i+j)/n), \quad \text{i.e.} \quad M = H,$$

and

$$g = \begin{bmatrix} g_0 \\ \vdots \\ g_{4n} \end{bmatrix}, \quad q = \begin{bmatrix} q_0 & \dots & q_{4n} \end{bmatrix}.$$

Since $M = H$ is a non-singular matrix, and $W = W' > 0$,

$$\min_{q:\ qHg=1} qHWHq = \min_{\delta:\ \delta' g=1} \delta' W \delta = \frac{1}{g' W^{-1} g}$$

for $g \neq 0$. Similarly,

$$\max_{g:\ qHg=1} g' HWHg = \frac{1}{qW^{-1}q'} \quad (q \neq 0).$$

Hence $\hat{\phi}_p(g)$ is maximized on the dominant (column) eigenvectors of $WHWH$, and $\hat{\phi}_d(q)$ is maximized on the dominant row eigenvectors of $WHWH$. Then the linear span of vectors v_i $(i = 1, \dots, r)$ should coincide with the linear span of functions $F(t)\bar{g}_i$, where $\bar{g}_i$ are the dominant right eigenvectors of $WHWH$, and the linear span of linear functionals u_i $(i = 1, \dots, r)$ should coincide with the linear span of functionals

$$Lx = \bar{q}_i \begin{bmatrix} x(0) \\ x(1/n) \\ \vdots \\ x(4n/n) \end{bmatrix},$$

where $\bar{q}_i$ are the dominant left eigenvectors of $WHWH$.

One convenient way to define $\bar{g}_i$, $\bar{q}_i$ is by $\bar{g}_i = S\phi_i$ and $\bar{q}_i = \phi' S^{-1}$, where ϕ_i are dominant orthonormalized eigenvectors of $S'HS$, where S is the result of Choleski factorization $W = SS'$.

Let

$$U = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}$$

be defined by

$$Ux = \bar{q} \begin{bmatrix} x(0) \\ \vdots \\ x(4n/n) \end{bmatrix},$$

where

$$\bar{q} = \begin{bmatrix} \bar{q}_1 \\ \vdots \\ \bar{q}_r \end{bmatrix} = \begin{bmatrix} \phi_1' \\ \vdots \\ \phi_r' \end{bmatrix} S^{-1}.$$

Let

$$V = [v_1 \vdots v_r] = F(t)\bar{g}D,$$

where

$$\bar{g} = [\bar{g}_1 \vdots \bar{g}_r] = S[\phi_1 \ \ldots \ \phi_r],$$

and D is a non-singular n-by-n matrix. Then

$$\begin{aligned} UB &= \bar{q}\bar{h}, \\ CV &= \bar{e}H\bar{g}D, \\ UV &= \bar{q}H\bar{g}D, \end{aligned}$$

$$\lim_{t \to 0^+} U(e^{At}V - V) = \bar{q}\Lambda H\bar{g}D,$$

where

$$\bar{e} = [1 \ 0 \ \ldots \ 0], \quad \bar{h} = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{4n-1} \\ h_{4n} \end{bmatrix}, \quad \Lambda = n \begin{bmatrix} -1 & 1 & 0 & & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ 0 & 0 & -1 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & -1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & -1 \end{bmatrix}.$$

When $r = 3n + 1$, the resulting system has transfer function

$$\hat{G}_{3n+1} = \bar{e}(sI - \Lambda)^{-1}\bar{h}.$$

(c) Implement the resulting model reduction algorithm in MATLAB and test the reduced models for $r \in \{5, 10\}$ and $N \in \{50, 500\}$.

The algorithm is implemented in `ps42_6242_2004.m`. It appears that $\hat{G}$ cannot provide a better approximation of G than the one given by $\hat{G}_{3n+1}$. In other words, the algorithm reduces to applying balanced truncation to a particular high order approximation of G (given by $\hat{G}_{3n+1}$).

Problem 4.3

The objective of this assignment is to analyze existence and uniqueness of solution in a particular class of moments matching problems.

Consider the following moments matching problem: given a sequence of $2n$ real numbers $(f_k)_{k=0}^{2n-1}$, find real polynomials p, q such that $\deg(q) = n$, $\deg(p) \leq n-1$, $q(0) \neq 0$, and f_k are the first $2n$ moments of $p(s)/q(s)$ at $s = 0$, i.e.

$$\frac{p(s)}{q(s)} = O(s^{2n}) + \sum_{k=0}^{2n-1} f_k s^k$$

for $s \to 0$.

(a) Show that there exists a unique non-negative $r \leq n$ and a pair of real polynomials

$$\bar{p}(s) = \bar{p}_0 + \bar{p}_1 s + \cdots + \bar{p}_{n-r-1} s^{n-r-1},$$
$$\bar{q}(s) = 1 + \bar{q}_1 s + \cdots + \bar{q}_{n-r-1} s^{n-r-1} + \bar{q}_{n-r} s^{n-r}$$

with no common zeros, such that the first $2n-r$ moments of $\bar{f}(s) = \bar{p}(s)/\bar{q}(s)$ at $s = 0$ are $f_0, f_1, \ldots, f_{2n-r-1}$.

This is, essentially, a special case of Theorem 7.2. Nevertheless, let us give a proof "from scratch". For

$$f(s) = \sum_{k=0}^{2n-1} f_k s^k,$$

consider the linear transformation mapping the coefficients of polynomials $\hat{p}, \hat{q}$ of order not exceeding $n-1$ and n respectively, into the first $2n$ moments of $\hat{p} - \hat{q}f$. This is a linear transformation of a $2n+1$-dimensional vector space into a $2n$-dimensional vector space. Hence, it maps a non-zero element into zero. In other words, there exist polynomials $\hat{p}_0, \hat{q}_0$, not simultaneously equal to zero, such that the first $2n$ coefficients of $\hat{p} - \hat{q}f$ are equal to zero, i.e.

$$\hat{p}(s) - \hat{q}(s)f(s) = s^{2n}e(s)$$

for some polynomial e.

Let d be the greatest common denominator of $\hat{p}$ and $\hat{q}$. Let r be the number of zeros of d at $s = 0$, so that $d(s) = s^r d_1(s)$, where $d_1(0) \neq 0$. Note that $r \leq n < 2n$, and hence the polynomial $\hat{q}(s)/d(s)$ has a non-zero value c at $s = 0$, because otherwise

$$\hat{p}(s)/d(s) = f(s)\hat{q}(s)/d(s) + s^{2n}e(s)/d(s)$$

also has a root at zero, which contradicts the definition of $d = d(s)$. Define $\bar{p} = c^{-1}\hat{p}_0/d$, $\bar{q} = c^{-1}\hat{q}_0/d$. Now $\bar{q}(0) = 1$, the order of $\bar{q}$ is not larger than $n - r$, the order of $\bar{p}$ is smaller than $n - r$, and

$$\bar{p}(s) - \bar{q}(s)f(s) = s^{2n-r}e_1(s)$$

for some polynomial e_1, which means that $\bar{p}(s)/\bar{q}(s)$ matches the first $2n-r$ moments $f_0, f_1, \ldots, f_{2n-r-1}$. This proves *existence* of $\bar{p}, \bar{q}$ with the stated properties.

To prove *uniqueness* of $\bar{p}, \bar{q}$, consider two pairs of such polynomials, $(\bar{p}^0, \bar{q}^0)$ and $(\bar{p}^1, \bar{q}^1)$, matching the first $2n - r_0$ and $2n - r_1$ moments, respectively. Then, for

$$\hat{p}^i(s) = s^{r_i}\bar{p}^i(s), \quad \hat{q}^i(s) = s^{r_i}\bar{q}^i(s),$$

we have

$$\hat{p}^i(s) - f(s)\hat{q}^i(s) = s^{2n}e^i(s),$$

and hence the polynomial

$$\delta(s) = \hat{p}^1(s)\hat{q}^2(s) - \hat{p}^2(s)\hat{q}^1(s),$$

of degree less than $2n$, has at least $2n$ roots at $s = 0$. This implies $\delta \equiv 0$, and hence

$$\bar{p}^1(s)\bar{q}^2(s) = \bar{p}_2(s)\bar{q}^1(s).$$

Since $\bar{p}^i$ and $\bar{q}^i$ do not have common roots, it follows that $q^0 = cq^1$ and $p^0 = cp^1$ for some constant $c \neq 0$. Since $\bar{q}^1(0) = \bar{q}^2(0) = 1$, we have $c = 1$.

(b) Let d be the maximal number of first moments of $\bar{f}(s)$ at $s = 0$ which are matching the numbers $f_0, f_1, \ldots, f_{2n-1}$. Let n_p, n_q be the degrees of $\bar{p}$ and $\bar{q}$ respectively. In terms of numbers r, d, n_p, n_q, give necessary and sufficient conditions for existence and (separately) uniqueness of solutions of the original moments matching problem.

A solution exists if and only if $d = 2n$ and $n_p < n_q$. The solution is unique if, in addition, $n_q = n$. These statements are a special case of Theorem 7.3.

(c) USE THE RESULT FROM (B) TO GENERATE REAL SEQUENCES $(f_k)_{k=0}^{2n-1}$ WITH $n = 3$ AND $f_k \neq 0$, SUCH THAT THE MOMENTS MATCHING PROBLEM HAS NO SOLUTION AND (SEPARATELY) HAS MANY SOLUTIONS.

To produce $\{f_k\}_{k=0}^{5}$ such that the moments matching problem has no solution, simply define f_k as moments of a ratio of polynomials of degree less than three which is not strictly proper. For example, $f_0 = 1$, $f_k = 0$ for $k > 0$ (the moments of $F(s) = 1/1$), or $f_0 = f_1 = f_3 = f_4 = f_5 = 0$, $f_2 = 1$ (the moments of $F(s) = s^2/1$).

To produce $\{f_k\}_{k=0}^{5}$ such that the moments matching problem has multiple solutions, simply define f_k as moments of a strictly proper transfer function of degree *less* than 3. For example, $f_k \equiv 1$ are the moments of $F(s) = 1/(1-s)$, but are also the moments of $F(s) = (1-s)/(1-s)^2$.