Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.242, Fall 2004: MODEL REDUCTION *

Problem set 5 solutions[1]

Problem 5.1

For each of the stetements below, state if it is true or false. For false statements, give a counterexample. For correct statements, give a *brief* sketch of a proof.

(a) H-Infinity norm of system with transfer function

$$H(s) = D + C(sI - A)^{-1}B$$

is not smaller than $|D|$.

True. Since $H(j\omega) \to D$ as $\omega \to \infty$, the supremum of $|H(j\omega)|$ is not smaller than $|D|$.

(b) If A, B, C, D are matrices of dimensions n-by-n, n-by-1, 1-by-n, and 1-

[1]Version of November 2, 2004.

BY-1 RESPECTIVELY, AND MATRICES

$$M_c = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}, \quad M_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

HAVE RANK $n-1$ THEN ORDER OF SYSTEM WITH TRANSFER FUNCTION

$$H(s) = D + C(sI - A)^{-1}B$$

EQUALS $n-1$.

False. For example, with $n = 3$,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad D = 0,$$

both matrices

$$M_c = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_o = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

have rank $n - 1 = 2$, but $H(s) = 1/s$ has order 1.

(c) IF $Aq = 0$ FOR SOME VECTOR $q \neq 0$ THEN

$$\lim_{s \to 0} sC(sI - A)^{-1}B = Cq.$$

False. For example, with $A = 0, B = C = 1$, $q = 2$, we have $Cq = 2$ but $sC(sI - A)^{-1}B \to 1$ as $s \to 0$.

(d) IF A IS A HURWITZ MATRIX, AND MATRIX V IS SUCH THAT $V'V = I$ THEN $V'AV$ IS A HURWITZ MATRIX AS WELL.

False. For example, if

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

then A is a Hurwitz matrix, and $V'V = 1$, but $V'AV = 0$ is not a Hurwitz matrix.

(e) IF A IS A HURWITZ MATRIX, THE PAIR (A, B) IS CONTROLLABLE, THE PAIR (C, A) IS OBSERVABLE, AND $W > 0$ IS A DIAGONAL MATRIX SUCH THAT

$$AW + WA' = -BB', \quad WA + A'W = -C'C,$$

THEN $A_{11} < 0$, WHERE A_{11} IS THE UPPER LEFT CORNER ELEMENT OF A.

False. for example, if

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then all conditions are satisfied, but $A_{11} = 0$.

(f) IF A PROPER RATIONAL TRANSFER FUNCTION $G = G(s)$ WITHOUT POLES IN THE CLOSED RIGHT HALF PLANE SATISFIES $|G(j\omega)| \leq 1$ FOR ALL $\omega \in \mathbf{R}$, ALL HANKEL SINGULAR NUMBERS OF G ARE NOT LARGER THAN 1.

True. Since G can be approximated by 0 (a zero order transfer function) with H-Infinity norm of the modeling error not larger than one, the largest Hankel singular number of G, which is a lower bound for such error, must be not larger than 1.

Problem 5.2

WHAT IS THE ORDER OF THE LTI SYSTEM WITH TRANSFER MATRIX

$$H(s) = \begin{bmatrix} 1/(s+1) & 1/(s+2) \\ 1/(s+3) & 1/(s+3) \end{bmatrix} ?$$

The system has order 3. Indeed, the system has state space model with

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = 0,$$

which is both controllable and observable.

Problem 5.3

KNOWING THAT $G(j\omega) = 1+j$ FOR $\omega = 1$, $G(0) = 1$, $G(-1) = 5$, AND $G(\infty) = 4$, WHAT IS THE BEST LOWER BOUND OF THE H-INFINITY NORM OF A RATIONAL FUNCTION G?

The best lower bound is 4. Indeed, since the points $0, j, \infty$ are on the "extended" imaginary axis, the H-Infinity norm cannot be smaller than any of the numbers $|1 + j| = \sqrt{2}$, 1, 4. On the other hand, transfer function

$$G(s) = \frac{16(s^2+1)(s+1)}{(s+2)^4} - \frac{5s(s^2+1)}{2(s+2)^4} + \frac{s(s+1)(41+38s)}{(s+2)^5} + \frac{4s(s^2+1)(s+1)}{(s+2)^4}$$

satisfies all interpolation constraints, and its H-Infinity norm is exactly 4.

Problem 5.4

FIND L2 GAIN OF THE SYSTEM WHICH MAPS SCALAR INPUTS $f(t)$ INTO OUTPUTS

$$y(t) = f(t-1)/(1 + t^2 + |f(t-1)|^2).$$

The L2 gain equals zero. To see this, note that the system is a series connection of delay by 1,

$$f(t) \mapsto g(t) = f(t-1)$$

and a memoryless system

$$g(t) \mapsto y(t) = g(t)/(1 + t^2 + |g(t)|^2).$$

Note that the delay has L2 gain 1. Also, since the maximal possible value of $|y(t)|/|g(t)|$ converges to zero as $t \to \infty$, L2 gain of the memoryless block equals zero. Hence the overall L2 gain is zero.

Problem 5.5

A, B, C, D ARE MATRICES OF DIMENSIONS n-BY-n, n-BY-1, 1-BY-n, AND 1-BY-1 RESPECTIVELY, SUCH THAT $CA^{-3}B = 1$ AND MATRIX UV, WHERE

$$U = \begin{bmatrix} C \\ CA^{-1} \end{bmatrix}, \quad V = \begin{bmatrix} B & A^{-1}B \end{bmatrix},$$

IS NOT SINGULAR. IS THIS INFORMATION SUFFICIENT TO FIND $\hat{C}\hat{A}^{-3}\hat{B}$, WHERE

$$\hat{C} = CV, \quad \hat{B} = UB, \quad \hat{A} = UAV \ ?$$

The information is not sufficient.

Problem 5.6

WHAT IS THE VALUE OF THE 7TH LARGEST HANKEL SINGULAR VALUE OF

$$H(s) = (s^2 - 2s + 2)^5/(s^2 + 2s + 2)^5 \ ?$$

Since H is a stable all-pass tranfer function of 10th order, all of its ten positive singular numbers are equal to 1.

Problem 5.7

A FIFTH ORDER TRANSFER FUNCTION $\hat{G}$ IS OBTAINED BY APPLYING THE STANDARD BALANCED TRUNCATION ALGORITHM TO A SEVENTH ORDER TRANSFER FUNCTION G WHICH HAS HANKEL SINGULAR NUMBERS $7, 6, 5, 4, 3, 2, 1$. WHAT ARE THE HANKEL SINGULAR NUMBERS OF $\hat{G}$? WHAT IS THE RANGE OF POSSIBLE VALUES OF $\|G-\hat{G}\|_\infty$?

The Hankel singular numbers of $\hat{G}$ are the 5 largest singular numbers of G, i.e. $7, 6, 5, 4, 3$. The range of possible values of $\|G - \hat{G}\|_\infty$ is from σ_6 to $2(\sigma_6 + \sigma_7)$, i.e. $[2, 6]$.