Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.242, Fall 2004: MODEL REDUCTION *

Problem set 6 solutions[1]

Problem 6.1

(a) FIND AN ANALYTICAL EXPRESSION FOR THE COEFFICIENTS $c_1, \dots, c_n$ OF THE LINEAR COMBINATION

$$\hat{G}_n(s) = \sum_{k=1}^{n} \frac{c_k}{s + 1/k},$$

WHICH MINIMIZES THE INTEGRAL

$$\|G - \hat{G}_n\|_{H2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega) - \hat{G}_n(j\omega)|^2 d\omega,$$

WHERE

$$G(s) = \frac{s^{1/3}}{s+1}.$$

For the optimal coefficients c_k, the scalar product of the error transfer function $\Delta(s) = G(s) - G_n(s)$ with each of the basis functions $1/(s + 1/k)$ must be zero.

Note that, for a strictly proper rational transfer function $H = H(s)$ with no poles in the closed right half plane, the scalar product $\langle H, H_a \rangle$ of H with $H_a(s) = 1/(s+a)$,

[1]Version of December 2, 2004

where $a > 0$, is given by $\langle H, H_a\rangle = H(a)$. Indeed, if $h = h(t)$ is the inverse Laplace transform of H, then the Parceval formulae yields

$$\langle H, H_a\rangle = \int_0^\infty e^{-at}h(t)dt = H(a).$$

Hence, the optimal coefficients satisfy

$$G(1/m) = \sum_{k=1}^{n} \frac{c_k}{1/m + 1/k}, \quad m = 1, 2, \ldots, n,$$

which yields $c = W^{-1}g$, where

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad g = \begin{bmatrix} \frac{1}{2} \\ \frac{2^{-1/3}}{1+1/2} \\ \vdots \\ \frac{n^{-1/3}}{1+1/n} \end{bmatrix}, \quad W = \begin{bmatrix} \frac{1}{1+1} & \frac{1}{1+1/2} & \cdots & \frac{1}{1+1/n} \\ \frac{1}{1+1/2} & \frac{1}{1/2+1/2} & & \\ \vdots & & \ddots & \\ \frac{1}{1+1/n} & & & \frac{1}{1/n+1/n} \end{bmatrix}.$$

(b) FOR $n = 1, 2, \ldots, 50$, USE MATLAB TO COMPUTE AND COMPARE $\|G - \hat{G}_n\|_{H2}$ AND $\|G - \hat{G}_n\|_\infty$.

Since, for the optimal c_k, functions $H_{1/k}(s) = 1/(s + 1/k)$ are orthogonal to $G - \hat{G}$ for $k = 1, 2, \ldots, n$, it follows that $G - \hat{G}$ is orthogonal to $\hat{G}$, and hence

$$\|G - \hat{G}\|_{H2}^2 = \|G\|_{H2}^2 - \|\hat{G}\|_{H2}^2.$$

Here

$$\|\hat{G}\|_{H2}^2 = c'Wc = g'W^{-1}g,$$

and $\|G\|_{H2}^2$ can be calculated as

$$\begin{aligned} \|G\|_{H2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^{2/3}d\omega}{\omega^2+1} \\ &= \frac{1}{\pi}\int_0^\infty \frac{\omega^{2/3}d\omega}{\omega^2+1} \\ &= \frac{3}{\pi}\int_0^\infty \frac{r^4dr}{r^6+1} \\ &= 3\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^4d\omega}{\omega^6+1} \\ &= 3\left\|\frac{s^2}{(s+1)(s^2+s+1)}\right\|_{H2}^2 \\ &= 1. \end{aligned}$$

The MATLAB code is given in `ps61_6242_2004.m`. Since matrix W is very poorly conditioned, a regularization approach is used. With $a_k = 1/k$, as suggested, the resulting approximations are very low quality. Performance of the algorithm gets better when pole locations $a_k = k$ are added.

Problem 6.2

(a) IS IT TRUE OR FALSE: IF $f: [0,1] \mapsto (-\infty, 0)$ IS CONVEX THEN $1/f$ IS CONVEX AS WELL?

False. For example, take $f(t) = t - 2$: then $1/f(0) = -1/2$, $1/f(1) = -1$, but $1/f(1/2) = -2/3 > 1/2(-1/2-1)$. In fact,it can be shown that, under the assumptions made, function $1/f$ is *concave*!

(b) IS IT TRUE OR FALSE: THE SUM OF TWO QUASI-CONVEX FUNCTIONS $f_1, f_2 : \Omega \mapsto \mathbf{R}$ IS ALWAYS QUASI-CONVEX?

False. For example, take $\Omega = \mathbf{R} = \{t\}$, $f_1(t) = -e^t$, $f_2(t) = -e^{-t}$. Both functions are monotonic, and hence quasi-convex. Nevertheless, their sum is not quasi-convex.

(c) FOR $m, n \in \{1, 2, 3, \dots\}$ LET $\Phi : \Omega_{n,m} \mapsto \mathbf{R}$ BE THE FUNCTION

$$\Phi_{n,m}(x) = \sum_{k=1}^{n} \left| \frac{p(k/n)}{q(k/n)} - y_k \right|,$$

WHERE

$$p(t) = p_0 + p_1 t + \dots + p_m t^m, \quad q(t) = 1 + q_1 t + \dots + q_m t^m,$$

AND $\Omega_{n,m} = \{x\}$ IS THE SET OF VECTORS

$$x = [p_0; p_1; \dots; p_m; q_1; q_2; \dots; q_m; y_1; y_2; \dots; y_n]$$

SUCH THAT $q(t) \neq 0$ FOR ALL $t \in [0, 1]$. FOR WHICH VALUES OF $m, n \in \{1, 2, 3, \dots\}$ IS $\Phi_{n,m}$ QUASI-CONVEX?

False. Take the two points $x^1, x^2 \in \Omega$, defined by

$$p^1(t) \equiv 1, \; q^1(t) \equiv 1 + t/2, \; y_k^1 = (1 + k/2n)^{-1},$$

and

$$p^2(t) \equiv 1,\ q^2(t) \equiv 1 - t/2,\ y^1_k = (1 - k/2n)^{-1},$$

respectively. Note that $\Phi(x^1) = \Phi(x^2) = 0$. However, for the middle point $x = 0.5(x^1 + x^2)$, defined by

$$p \equiv 1,\ q(t) \equiv 1,\ y_k = (1 - k^2/4n^2),$$

produces $\Phi(x) > 0$. Hence the level set $\{x :\ \Phi(x) \leq 0\}$ is not convex.

Problem 6.3

(a) FOR EVERY $n \in \{1, 2, \dots\}$ AND $\epsilon > 0$, DEFINE $N = N(n)$ AND AN AFFINE SYMMETRIC MATRIX FUNCTION $A = A(x)$ OF VECTOR

$$x = [b_0; b_1; \dots; b_n; a_0; a_1; \dots; a_{n-1}; y_1; \dots; y_N]$$

SUCH THAT, GIVEN $b_0, \dots, b_n$ AND $a_0, \dots, a_{n-1}$, THE INEQUALITY $A(x) > 0$ HAS A SOLUTION WITH RESPECT TO $y_1, \dots, y_N$ IF AND ONLY IF

$$b(\omega^2) > 0, \quad a(\omega^2) > 0, \quad \frac{b(\omega^2)}{a(\omega^2)} < 1 + \epsilon \ \ \forall\ \omega \in \mathbf{R},$$

$$\frac{b(\omega^2)}{a(\omega^2)} > 1 - \epsilon\ \forall\ |\omega| \leq 1,$$

AND

$$\frac{b(\omega^2)}{a(\omega^2)} < \epsilon\ \forall\ |\omega| \geq 1 + \epsilon,$$

WHERE

$$a(\theta) = a_0 + a_1\theta + \dots + a_{n-1}\theta^{n-1} + \theta^n, \quad b(\theta) = b_0\theta + b_1\theta + \dots + b_n\theta^n.$$

First, note that the formulaion has to be modified in order for the problem to be solvable. Indeed, the set of coefficients b_k for which $b(\omega^2) > 0$ for all $\omega \in \mathbf{R}$ is *not* an open set. For example, the set of all pairs $(b_0, b_1) \in \mathbf{R}^2$ such that $b_0 + b_1\omega^2 > 0$ for all ω is defined by the inequalities $b_0 > 0$, $b_1 \geq 0$. Since the set of x such that $A(x) > 0$ is always open, and linear projection of an open set is open as well, the construction is not possible, as requested. However, if one interprets positivity of a polynomial

$$h(\omega) = h_0 + h_1\omega^2 + \cdots + h_n\omega^{2n} \tag{6.1}$$

in a strict sense, meaning that the values of the ratio $h(\omega)/(1+\omega^2)^n$ with ω ranging over $\mathbf{R}$ are separated from zero, the desired $A = A(x)$ can be constructed.

Indeed, the polynomial (6.1) can be represented by

$$\frac{h(\omega)}{(1+\omega^2)^n} = G(j\omega)'HG(j\omega),$$

where

$$H = \begin{bmatrix} h_0 & 0 & \dots & 0 \\ 0 & h_1 & & \\ \vdots & & \ddots & \\ 0 & & & h_n \end{bmatrix}, \quad G(s) = \frac{1}{(s+1)^n}\begin{bmatrix} 1 \\ s \\ \vdots \\ s^n \end{bmatrix}.$$

Hence, according to the KYP lemma, strict positivity of $h(\omega)/(1+\omega^2)^n$ is equivalent to the existence of a symmetric matrix $Q = Q'$ such that the quadratic form

$$2x'Q(A_Gx + B_Gf) + (C_Gx + D_Gf)'H(C_Gx + D_Gf),$$

where matrices A_G, B_G, C_G, D_G define a minimal state space model of G, is strictly positive definite. Hence, strict positivity of $h(\omega)$ is equivalent to solvability of the semidefinite program

$$L_{G1}QL'_{G2} + L_{G2}QL'_{G1} + L_{G3}HL'_{G3} > 0,$$

where

$$L_{G1} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \ L_{G2} = \begin{bmatrix} A'_G \\ B'_G \end{bmatrix}, \ L_{G3} = \begin{bmatrix} C'_G \\ D'_G \end{bmatrix},$$

with respect to $Q = Q'$.

This observation allows one to convert the inequalities

$$a(\omega^2) > 0, \ b(\omega^2) > 0, \ (1+\epsilon)a(\omega^2) - b(\omega^2) > 0 \ \ \forall\, \omega \in \mathbf{R},$$

understood in a strict sense, into linear matrix inequalities with respect to the coefficients of a, b and the auxiliary symmetric n-by-n matrices Q_1, Q_2, Q_3. In all three cases, matrices L_{G1}, L_{G2}, L_{G3} are the same, and the diagonal coefficients of H depend on the coefficients of a, b differently:

(a) for inequality $a(\omega^2) > 0$, use $h_k = a_k$ for $k = 0, 1, \ldots, n-1$, and $h_n = 1$;

(b) for inequality $b(\omega^2) > 0$, use $h_k = b_k$ for $k = 0, 1, \ldots, n$;

(c) for inequality $(1+\epsilon)a(\omega^2) - b(\omega^2) > 0$, use $h_k = (1+\epsilon)a_k - b_k$ for $k = 0, 1, \ldots, n-1$, and $h_n = 1 + \epsilon - b_n$.

To convert the conditional inequalities

$$b(\omega^2) - (1-\epsilon)a(\omega^2) > 0 \ (|\omega| \le 1), \quad \epsilon a(\omega^2) - b(\omega^2) > 0 \ (|\omega| \ge 1+\epsilon), \tag{6.2}$$

note that positivity of the polynomial in (6.1) for $|\omega| \le 1$ is equivalent to positivity of the rational function

$$\tilde{h}(j\omega) = \sum_{k=0}^{n} \frac{h_k}{(1+\omega^2)^k}$$

for all ω. Similarly, positivity of $h(\omega)$ for $|\omega| \ge r > 0$ is equivalent to positivity of the rational function

$$\sum_{k=0}^{n} \frac{h_{n-k}}{r^{2k}(1+\omega^2)^k}$$

for all ω.

Since

$$\sum_{k=0}^{n} \frac{h_k}{(1+\omega^2)^k} = F(j\omega)' H F(j\omega),$$

where

$$F(s) = \begin{bmatrix} 1 \\ \frac{1}{s+1} \\ \frac{1}{(s+1)^2} \\ \vdots \\ \frac{1}{(s+1)^n} \end{bmatrix},$$

$\tilde{h}(j\omega) > 0$ for all ω if and only if there exists a symmetric matrix $P = P'$ such that the quadratic form

$$2x'P(A_F x + B_F f) + (C_F x + D_F f)' H (C_F x + D_F f)$$

is strictly positive definite. Hence, strict positive definiteness of $\tilde{h}(\omega)$ is equivalent to solvability of the semidefinite program

$$L_{F1} P L_{F2}' + L_{F2} P L_{F1}' + L_{F3} H L_{F3}' > 0,$$

where

$$L_{F1} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \; L_{F2} = \begin{bmatrix} A_F' \\ B_F' \end{bmatrix}, \; L_{F3} = \begin{bmatrix} C_F' \\ D_F' \end{bmatrix},$$

with respect to $P = P'$.

This observation allows one to convert the inequalities (6.2), understood in a strict sense, into linear matrix inequalities with respect to the coefficients of a, b and the auxiliary symmetric n-by-n matrices P_1, P_2. In both cases, matrices L_{F1}, L_{F2}, L_{F3} are the same, and the diagonal coefficients of H depend on the coefficients of a, b differently:

(a) for inequality $b(\omega^2) - (1-\epsilon)a(\omega^2) > 0$, where $|\omega| \leq 1$, use $h_k = b_k - (1-\epsilon)a_k$ for $k = 0, 1, \dots, n-1$, and $h_n = b_n - 1 + \epsilon$;

(c) for inequality $\epsilon a(\omega^2) - b(\omega^2) > 0$, where $|\omega| \geq 1+\epsilon$, use $h_k = r^{-2k}(\epsilon a_{n-k} - b_{n-k})$ for $k = 1, \dots, n$, and $h_0 = \epsilon - b_n$.

(b) Use the result from (a) and a semidefinite program solver (see section 4 of Lecture 9 for some options) to write a MATLAB code for designing high quality low pass filters, in the form of a stable n-th order transfer function G such that

$$\|G\|_\infty^2 < 1 + \epsilon, \quad |G(j\omega)|^2 > 1 - \epsilon \;\forall\; \omega \in [0, 1], \quad |G(j\omega)|^2 < \epsilon \;\forall\; \omega \in [1+\epsilon, \infty),$$

where $\epsilon > 0$ is a given small parameter.

An implementation of the algorithm is given in `ps63_6242_2004.m`. It is common for the LMI optimization algorithm to terminate without solving the LMI's exactly, but still provide a good quality filter. For example, for $n = 10$, the problem has a solution with $\epsilon = 0.1$.