Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.242, Fall 2004: MODEL REDUCTION *

Take-home test 1[1]

Test 1 papers are due by 4pm on November 5, 2004, in Alex's office. Please, **no cooperation** regarding test problems!

Problem Q1.1

For each of the stetements below, state if it is true or false. For false statements, give a counterexample. For correct statements, give a *brief* sketch of a proof.

(a) If the pair (A, B) is controllable, and $A + A' = -BB'$ then A is a Hurwitz matrix.

(b) If $q(s)$ is a Hurwitz polynomial of order n, and p_1, p_2, p_3, p_4 are polynomials of order n then system with transfer matrix

$$G(s) = \begin{bmatrix} p_1(s)/q(s) & p_2(s)/q(s) \\ p_3(s)/q(s) & p_4(s)/q(s) \end{bmatrix}$$

has order not larger than n.

(c) If A is a Hurwitz matrix, the columns of a non-square matrix V are (some) eigenvectors of A , and $V'V$ is not a singular matrix then $\hat{A} = (V'V)^{-1}V'AV$ is a Hurwitz matrix as well.

[1]Version of November 3, 2004. Due November 5, 2004, 4pm.

(d) If A, B, C are matrices of dimensions n-by-n, n-by-1, and 1-by-n respectively, and A is a Hurwitz matrix, then there exist a non-singular n-by-n matrix S such that, for

$$\hat{A} = S^{-1}AS, \quad \hat{B} = S^{-1}B, \quad \hat{C} = CS,$$

the solutions W_o and W_c of

$$\hat{A}W_c + W_c\hat{A}' = -\hat{B}\hat{B}', \quad W_o\hat{A} + \hat{A}'W_o = -\hat{C}'\hat{C},$$

are equal.

(e) If a proper rational transfer function $G = G(s)$ without poles in the closed right half plane satisfies

$$|G(j\omega) - 1/(1 + j\omega)| \leq 1$$

for all $\omega \in \mathbf{R}$, then the first Hankel singular number of G is not larger than 1.5.

Problem Q1.2

For all $a > 0$, find Hankel singular numbers of the stable LTI system with transfer matrix

$$G(s) = \begin{bmatrix} 1/(s+a) & 0 \\ 1 & 1/(s+a) \end{bmatrix}.$$

Problem Q1.3

For all values of $a \in \mathbf{R}$, find L2 gain of the system which maps scalar inputs $f(t)$ into outputs

$$y(t) = \sin(f(t - a)).$$

Problem Q1.4

A, B, C are matrices of dimensions n-by-n, n-by-1, and 1-by-n respectively, (A is a Hurwitz matrix),

$$U = \begin{bmatrix} C(I - A)^{-1} \\ C(I - A)^{-2} \end{bmatrix}, \quad V = \begin{bmatrix} (I - A)^{-1}B & (I - A)^{-2}B \end{bmatrix},$$

matrix UV is not singular, and

$$CV(sI - (UV)^{-1}UAV)^{-1}(UV)^{-1}UB = 1/s^2.$$

Find $C(I - A)^{-4}B$.

Problem Q1.5

A, B, C are matrices of dimensions n-by-n, n-by-1, and 1-by-n respectively, and A is a Hurwitz matrix. It is known that

$$C(I - A)^{-1}B = 0, \ C(I - A)^{-2}B = -1, C(I - A)^{-3}B = 1.$$

Find positive lower bounds for Hankel singular numbers $\sigma_1(G)$ and $\sigma_2(G)$, where $G(s) = C(sI - A)^{-1}B$.