Massachusetts Institute of Technology

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6.242, Fall 2004: MODEL REDUCTION *

Take-home test 1 solutions[1]

Problem Q1.1

For each of the stetements below, state if it is true or false. For false statements, give a counterexample. For correct statements, give a *brief* sketch of a proof.

(a) If the pair (A, B) is controllable, and $A + A' = -BB'$ then A is a Hurwitz matrix.

True.

Assume the contrary, i.e. that there exists $s \in \mathbf{C}$ such that $\text{Re}(s) \geq 0$ and $\det(sI - A) = 0$. Let $f \in \mathbf{C}^n$, $f \neq 0$ be the corresponding eigenvector. Then $Af = sf$, $f'A' = \bar{s}f'$, and hence multiplication of $A + A' = -BB'$ by f' on the left and f on the right yields

$$(s + \bar{s})f'f = -f'BB'f, \quad \text{i.e.} \quad 2\text{Re}(s)|f|^2 = -|B'f|^2.$$

Since $\text{Re}(s) \geq 0$ and $|f|^2 > 0$, we conclude that $B'f = 0$, which contradicts the controllability assumption.

[1] Version of November 8, 2004.

(b) If $q(s)$ is a Hurwitz polynomial of order n, and p_1, p_2, p_3, p_4 are polynomials of order n then system with transfer matrix

$$G(s) = \begin{bmatrix} p_1(s)/q(s) & p_2(s)/q(s) \\ p_3(s)/q(s) & p_4(s)/q(s) \end{bmatrix}$$

has order not larger than n.

False.

For example, $q(s) = s+1$, $p_1(s) = p_4(s) = s-1$, $p_2(s) = p_3(s) = s+1$ yields $G(s)$ of order 2.

(c) If A is a Hurwitz matrix, the columns of a non-square matrix V are (some) eigenvectors of A , and $V'V$ is not a singular matrix then $\hat{A} = (V'V)^{-1}V'AV$ is a Hurwitz matrix as well.

True.

By assumption, $AV = VD$, where D is a diagonal matrix with eigenvalues of A on the diagonal. Hence $\hat{A} = D$ is a Hurwitz matrix.

(d) If A, B, C are matrices of dimensions n-by-n, n-by-1, and 1-by-n respectively, and A is a Hurwitz matrix, then there exist a non-singular n-by-n matrix S such that, for

$$\hat{A} = S^{-1}AS, \quad \hat{B} = S^{-1}B, \quad \hat{C} = CS,$$

the solutions W_o and W_c of

$$\hat{A}W_c + W_c\hat{A}' = -\hat{B}\hat{B}', \quad W_o\hat{A} + \hat{A}'W_o = -\hat{C}'\hat{C},$$

are equal.

False.

For example, system with $n = 1$, $A = -1$, $B = 1$, $C = 0$ is controllable but not observable. Hence, even after a change of coordinates, its observability Gramian will be zero, while its controllability Gramian will be positive.

(e) IF A PROPER RATIONAL TRANSFER FUNCTION $G = G(s)$ WITHOUT POLES IN THE CLOSED RIGHT HALF PLANE SATISFIES

$$|G(j\omega) - 1/(1 + j\omega)| \le 1$$

FOR ALL $\omega \in \mathbf{R}$, THEN THE FIRST HANKEL SINGULAR NUMBER OF G IS NOT LARGER THAN 1.5.

True.

Since

$$\left\| \frac{1}{s+1} - \frac{1}{2} \right\|_\infty = \left\| \frac{1}{2} \frac{1-s}{1+s} \right\|_\infty = 0.5,$$

we have

$$\left\| G(s) - \frac{1}{2} \right\|_\infty = \left\| G(s) - \frac{1}{s+1} + \frac{1}{s+1} - \frac{1}{2} \right\|_\infty \le 1 + 0.5 = 1.5.$$

Problem Q1.2

FOR ALL $a > 0$, FIND HANKEL SINGULAR NUMBERS OF THE STABLE LTI SYSTEM WITH TRANSFER MATRIX

$$G(s) = \begin{bmatrix} 1/(s+a) & 0 \\ 1 & 1/(s+a) \end{bmatrix}.$$

Answer: $\sigma_1(G) = \sigma_2(G) = 1/2a$.
Indeed, a state space realization of the system is given by

$$A = -aI_2,\ B = I_2,\ C = I_2,\ D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

For this realization, $W_c = W_o = (1/2a)I_2$. Hence, the only non-zero Hankel singular values of G are $\sigma_1(G) = \sigma_2(G) = 1/2a$.

Problem Q1.3

FOR ALL VALUES OF $a \in \mathbf{R}$, FIND L2 GAIN OF THE SYSTEM WHICH MAPS SCALAR INPUTS $f(t)$ INTO OUTPUTS

$$y(t) = \sin(f(t-a)).$$

Answer: the gain equals 1 for all $a \in \mathbf{R}$.

Indeed, since $|\sin(h)| \le |h|$ for all $h \in \mathbf{R}$, we have $|y(t)| \le |f(t-a)|$ for all t. In addition, since $|\sin(h)| \le 1$ for all $h \in \mathbf{R}$, we have $|y(t)| \le 1$ for all $t \in \mathbf{R}$.

Now, for $a \ge 0$, we have

$$\int_0^T \{|f(t)|^2 - |y(t)|^2\}dt =$$

$$\int_0^T |f(t)|^2 dt - \int_{-a}^{T-a} |f(t)|^2 \ge$$

$$-\int_{-a}^{0} |f(t)|^2 dt,$$

which is bounded from below as $T \to +\infty$. For $a < 0$ we have

$$\int_0^T \{|f(t)|^2 - |y(t)|^2\}dt =$$

$$\int_0^T |f(t)|^2 dt - \int_{-a}^{T-a} |f(t)|^2 \ge$$

$$-\int_{T}^{T-a} dt = -a,$$

which is also bounded from below as $T \to +\infty$. Hence the L2 gain is not larger than 1. On the other hand, input $f(t) \equiv \delta =$ const produces system response $y(t) \equiv \sin(\delta) =$ const. Since $\sin(\delta)/\delta \to 1$ as $\delta \to 0$, $\delta \ne 0$, the L2 gain is at least as large as 1.

Problem Q1.4

A, B, C ARE MATRICES OF DIMENSIONS n-BY-n, n-BY-1, AND 1-BY-n RESPECTIVELY, (A IS A HURWITZ MATRIX),

$$U = \begin{bmatrix} C(I-A)^{-1} \\ C(I-A)^{-2} \end{bmatrix}, \quad V = \begin{bmatrix} (I-A)^{-1}B & (I-A)^{-2}B \end{bmatrix},$$

MATRIX UV IS NOT SINGULAR, AND

$$CV(sI - (UV)^{-1}UAV)^{-1}(UV)^{-1}UB = 1/s^2.$$

FIND $C(I-A)^{-4}B$.

Answer: $C(I-A)^{-4}B = 4$.
Indeed,

$$\hat{G}(s) = CV(sI - (UV)^{-1}UAV)^{-1}(UV)^{-1}UB = 1/s^2$$

is the reduced model of

$$G(s) = C(sI - A)^{-1}B$$

obtained using a projection method, in such a way that the linear span of the columns of V includes vectors $(sI-A)^{-k}B$ for $s=1$, $k=1,2$, and the linear span of the rows of U includes $C(sI-A)^{-k}$ for $s=1$, $k=1,2$. Hence the first 4 moments of G and $\hat{G}$ at $s=1$ must be equal. Therefore, $C(I-A)^{-4}B$, as the 4-th moment of G at $s=1$ times -1, must be equal to minus the 4-th moment of $\hat{G}(s) = 1/s^2$ at $s=1$, i.e. $C(I-A)^{-4}B = -g_3$, where

$$\frac{1}{s^2} = g_0 + g_1(s-1) + g_2(s-1)^2 + g_3(s-1)^3 + O((s-1)^4) \quad \text{as} \quad s \to 1.$$

Since

$$\frac{1}{s^2} = (1-(s-1)+(s-1)^2-(s-1)^3+\dots)^2 = 1-2(s-1)+3(s-1)^2-4(s-1)^3+\dots,$$

we have $C(I-A)^{-4}B = 4$.

Problem Q1.5

A, B, C ARE MATRICES OF DIMENSIONS n-BY-n, n-BY-1, AND 1-BY-n RESPECTIVELY, AND A IS A HURWITZ MATRIX. IT IS KNOWN THAT

$$C(I-A)^{-1}B = 0, \;\; C(I-A)^{-2}B = -1, C(I-A)^{-3}B = 1.$$

FIND POSITIVE LOWER BOUNDS FOR HANKEL SINGULAR NUMBERS $\sigma_1(G)$ AND $\sigma_2(G)$, WHERE $G(s) = C(sI-A)^{-1}B$.

Answer: $\sigma_1(G) \geq 2$, $\sigma_2(G) \geq 1$ (though better lower bounds can be found).
To get the lower bound for $\sigma_1(G)$, use Theorem 5.5 from lecture notes with $s=1$, which implies that

$$W_c \geq W_c^- = 2(I-A)^{-1}BB'(I-A')^{-1}, \;\; W_o \geq W_o^- = 2(I-A')^{-1}C'C(I-A)^{-1}.$$

Hence

$$\sigma_1(G) \geq \lambda_{\max}(W_c^- W_o^-)^{1/2} = |2C(I-A)^{-2}B| = 2.$$

Getting a lower bound for $\sigma_2(G)$ is more difficult. Let

$$H_0(s) = \sqrt{2}\frac{G(s)-G(1)}{s-1} = \sum_{k=0}^{\infty} h_k \frac{\sqrt{2}(s-1)^k}{(s+1)^{k+1}}$$

be the expansion of the Laplace transform of the causal part of system response to input $f_0(t) = \sqrt{2}e^t u(-t)$ (Laplace transform $\sqrt{2}/(s-1)$. Note that

$$h_0 = -C(I-A)^{-2}B = 1, \quad h_1 = 2C(I-A)^{-2}B = 2.$$

System response $h_1(t)$ to anti-causal input $f_1(t)$, defined by its Laplace transform $F_1(s) = \sqrt{2}(s+1)/(s-1)^2$, is given by

$$H_1(s) = \sum_{k=0}^{\infty} h_{k+1} \frac{\sqrt{2}(s-1)^k}{(s+1)^{k+1}}.$$

Since $\sigma_2(G)$ is not smaller than the minimal (over $\alpha \in \mathbf{R}$) value of

$$\sum_{k=0}^{\infty} |h_k \cos(\alpha) + h_{k+1}\sin(\alpha)|^2,$$

and

$$\sum_{k=0}^{\infty} |h_k|^2 < \infty,$$

a simple calculation shows that $\sigma_2(G) \geq 1$.