Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.242, Fall 2004: MODEL REDUCTION *

Take-home test 2 solutions[1]

Problem Q2.1

Construct a strictly proper rational transfer function $G = G(s)$ with no poles in the closed right half plane for which there exists a unique first order transfer function $\hat{G} = \hat{G}(s)$ such that $G(1) = \hat{G}(1)$, and $G(j) = \hat{G}(j)$ ($j = \sqrt{-1}$), and this $\hat{G}$ has a pole with positive real part. (As usually, G and $\hat{G}$ have *real* coefficients.)

One such G is given by

$$G(s) = \frac{1}{5}\frac{2s-1}{s62+s+1} + \frac{56}{15}\frac{1+s^2}{(1+s)^3}.$$

According to theorem 7.3, if a proper first order transfer function $\hat{G}$ matching three moments of G exists, then it is a unique transfer function with this property. Hence, to construct an example, pick an arbitrary $\hat{G}$ (strictly proper, first order, unstable) at the beginning, and then find a stable strictly proper transfer function G of arbitrary order to match the values of $\hat{G}$ at $s = 1$ and $s = j$.

To keep calculations simple, use $\hat{G}(s) = 1/(2-s)$, and then search for G in the form

$$G(s) = \frac{c_1 s + c_0}{s^2+s+1} + c_2\frac{1+s^2}{(1+s)^3}.$$

[1] Version of December 8, 2004.

(It was necessary to have at least three free coefficients; the denominator s^2+s+1 was chosen because its value j at $s=j$ is “simple”, the numerator $1+s^2$ was chosen to be zero at $s=j$.) Matching the values at $s=j$ requires $c_1=2/5$ and $c_0=-1/5$. Taking this into account, matching the values at $s=1$ requires $c_2=56/15$.

Problem Q2.2

PROPER TRANSFER FUNCTION $\Delta=\Delta(s)$ WITH NO POLES IN THE CLOSED RIGHT HALF PLANE IS SUCH THAT $\Delta(1)=a$, WHERE $a\in\mathbf{R}$ IS A PARAMETER. WHAT IS THE MINIMAL POSSIBLE VALUE OF THE INTEGRAL

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\Delta(j\omega)|^2 d\omega}{1+\omega^2}\quad ?$$

The minimal value of the integral equals $a^2/2$.

It is easy to see that this value can be achieved using a constant $\Delta(s)\equiv a$. On the other hand, for $F(s)=\Delta(s)/(s+1)$, equality $F(1)=a/2$ is satisfied. Hence, for the impulse response $f=f(t)$ of transfer function $F=F(s)$, the Cauchy inequality yields

$$\frac{a^2}{4}=\left(\int_0^{\infty}f(t)e^{-t}dt\right)^2\leq$$

$$\leq\int_0^{\infty}|f(t)|^2dt\cdot\int_0^{\infty}e^{-2t}dt=\frac{1}{2}\int_0^{\infty}|f(t)|^2dt.$$

Therefore

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\Delta(j\omega)|^2 d\omega}{1+\omega^2}=\int_0^{\infty}|f(t)|^2dt\geq\frac{a^2}{2}.$$

Problem Q2.3

AMONG ALL PROPER RATIONAL TRANSFER FUNCTIONS $G=G(s)$ WITH NOT MORE THAN ONE POLE IN THE LEFT HALF PLANE FIND THE ONE FOR WHICH $\|G(s)-1/(s^2+s+1)\|_\infty$ IS MINIMAL.

In the original version, this problem was formulated with the word “right” instead of “left”, which led to a “trivial” solution $G(s)=1/(s^2+s+1)$.

Here is a solution of the more interesting version, where the answer is

$$G(s) = \frac{\frac{1-\sqrt{5}}{4}s + \frac{1+\sqrt{5}}{4}}{s + \frac{3-\sqrt{5}}{\sqrt{5}-1}} \approx \frac{-0.309s + 0.809}{s + 0.618}.$$

As stated, the problem relates to the proof of the AAK theorem, in the special case when a Hankel optimal $(n-1)$-st order approximation $\hat{H}$ of an n-th order transfer function H is sought ($H(s) = 1/(s^2+s+1)$ and $n = 2$ in this problem). From the proof of the AAK theorem, it is known that a Hankel optimal approximation $\hat{H}$ of order 1 can be constructed in such a way that $\hat{H} - H$ equals a constant times an all-pass transfer function. Therefore, the optimal G can be found in the form $G = \hat{H}$.

To actually calculate $\hat{H}$, assume it is of a general form

$$\hat{H}(s) = \frac{b_1 s + b_0}{s + a},$$

and find b_1, b_0, a, σ from the identity

$$\frac{b_1 s + b_0}{s + a} \frac{1}{s^2 + s + 1} = \sigma \frac{(s - a)(s^2 - s + 1)}{(s + a)(s^2 + s + 1)}.$$

Comparing the coefficients at both sides of the equality yields a system of equations

$$\begin{aligned} b_1 &= \sigma, \\ b_1 + b_0 &= -(1 + a)\sigma, \\ b_1 + b_0 - 1 &= (1 + a)\sigma, \\ b_0 - a &= -a\sigma. \end{aligned}$$

Solving it yields two solutions. The one with the smallest absolute value of σ is given by

$$\sigma = b_1 = \frac{1 - \sqrt{5}}{4}, \quad b_0 = \frac{1 + \sqrt{5}}{4}, \quad a = \frac{3 - \sqrt{5}}{\sqrt{5} - 1}.$$

Problem Q2.4

G is a 100-th order rational transfer function with no poles in the right half plane, such that $|G(j\omega)| = 1$ for all $\omega \in \mathbf{R}$. Find all possible values of the 10-th Hankel singular number of $\Delta(s) = G(s) - 1/(s + 1)$. Explain your answer.

The answer is $\sigma_{10}(\Delta) = 1$ (and, actually, $\sigma_k(\Delta) = 1$ for $k = 1, 2, \ldots, 99$).

To prove this, take into account the following general remark: if G_1 and G_2 are stable transfer functions, and the order of $G_1 - G_2$ is 1, then $\sigma_k(G_1) \geq \sigma_{k+1}(G_2)$ for all k. Indeed, if $\sigma_k(G_1) < \sigma_{k+1}(G_2)$ then, according to the AAK theorem, there exists $\delta = \delta(s)$ of order less than k such that $\sigma_1(G_1 - \delta) < \sigma_{k+1}(G_2)$. Equivalently,

$$\sigma_1(G_2 - (G_2 - G_1 + \delta)) < \sigma_{k+1}(G_2),$$

which is impossible since $G_2 - G_1 + \delta$ has order less than $k + 1$.

Note that G is an all-pass stable transfer function of order 100, and hence $\sigma_k(G) = 1$ for $k = 1, 2, \ldots, 100$. Applying the "general remark" with $G_1 = \Delta$, $G_2 = G$, and $k = 10$, yields $\sigma_{10}(\Delta) \geq \sigma_{11}(G) = 1$. Applying the "general remark" with $G_1 = G$, $G_2 = \Delta$, and $k = 9$, yields $1 = \sigma_9(G) \geq \sigma_{10}(\Delta)$. Hence $\sigma_{10}(\Delta) = 1$.

Problem Q2.5

Find a positive integer n and an affine symmetric matrix-valued function $\alpha = \alpha(z)$ of a real vector parameter $z \in \mathbf{R}^{n+3}$, such that the conditions $a_0 > 0$ and

$$\sup_{\omega \in \mathbf{R}} \left| \frac{b_0 + b_1\omega^2}{a_0 + \omega^2} - \frac{1}{(\omega^2 + 1)^2} \right| < \frac{1}{2}$$

are jointly satisfied if and only if there exist real numbers $y_1, \ldots, y_n$ such that $\alpha(z) > 0$ for

$$z = \begin{bmatrix} a_0 \\ b_0 \\ b_1 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

One possible solution has $n = 12$ and

$$\alpha(z) = \begin{bmatrix} L_2' P_+ L_1 + L_1' P_+ L_2 + S_0 + S_1 & 0 \\ 0 & L_2' P_- L_1 + L_1' P_- L_2 + S_0 - S_1 \end{bmatrix},$$

where

$$P_+ = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_2 & y_4 & y_5 \\ y_3 & y_5 & y_6 \end{bmatrix}, \quad P_- = \begin{bmatrix} y_7 & y_8 & y_9 \\ y_8 & y_{10} & y_{11} \\ y_9 & y_{11} & y_{12} \end{bmatrix},$$

$$L_1 = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$S_0 = \begin{bmatrix} \frac{a_0+1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad S_1 = \begin{bmatrix} b_0+b_1 & 0 & 0 & -b_1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1-a_0 & 0 \\ -b_1 & 0 & 0 & b_1 \end{bmatrix}.$$

This solution can be explained using the KYP Lemma in the following way. First, note that the original inequality, after multiplication by $(a_0+\omega^2)/(\omega^2+1)$, can be re-written in the equivalent form

$$\left| \frac{b_0+b_1\omega^2}{\omega^2+1} - \frac{a_0+\omega^2}{(\omega^2+1)^3} \right| < \frac{1}{2}\frac{a_0+\omega^2}{\omega^2+1}.$$

This is equivalent to a pair of inequalities

$$\frac{1}{2}\frac{a_0+\omega^2}{\omega^2+1} \pm \left[\frac{b_0+b_1\omega^2}{\omega^2+1} - \frac{a_0+\omega^2}{(\omega^2+1)^3} \right] > 0,$$

which are further equivalent to positive definiteness of Hermitean forms

$$\sigma_\pm(x,f) = \sigma_0(x,f) \pm \sigma_1(x,f),$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \sigma_0(x,f) = \frac{a_0}{2}|x_1|^2 + \frac{1}{2}|f-x_1|^2,$$

$$\sigma_1(x,f) = b_0|x_1|^2 + b_1|f-x_1|^2 - a_0|x_3|^2 - |x_2-x_3|^2,$$

on the subspaces defined by equalities

$$j\omega L_2 \begin{bmatrix} x \\ f \end{bmatrix} = L_1 \begin{bmatrix} x \\ f \end{bmatrix},$$

i.e.

$$\begin{aligned} j\omega x_1 &= -x_1 + f, \\ j\omega x_2 &= -x_2 + x_1, \\ j\omega x_3 &= -x_3 + x_2, \end{aligned}$$

for all $\omega \in \mathbf{R}$. According to the KYP Lemma, this is equivalent to existence of symmetric 3-by-3 matrices $P_\pm$ such that $\alpha(z) > 0$ (to see this, note that S_0 and S_1 are the matrices of σ_0 and σ_1 respectively).