Chapter 1

Introduction

The main objective of these lectures is to provide a reasonably complete but compact introduction to *robust* design and analysis of linear time invariant (LTI) feedback control for systems which are *linearizable*, i.e. can be approximated well by finite order LTI models. The modern approach, enabled by efficient computer-aided optimization algorithms, seeks to utilize rigorous treatment of design specifications, combined with careful accounting of model uncertainty and approximation errors. Typically, the need to use robust control arises when working with applications which have a "complexity" element, manifesting itself by the presence of non-standrad models (nonlinear, uncertain, time-varying, distributed), multiplicity of sensors and actuators, or when optimization of a quality factor is the main concern.

The lectures will explain basic principles, mathematical results, and numerical implementation strategies of LTI feedback design, including use of L2 gains or Integral Quadratic Constraints for quantifying quality of linear approximations and feedback system performance, use of generalized small gain conditions and convex optimization in analysis of robustness to modeling errors and uncertain parameters, and use of Schur decomposition for explicit optimization of linear feedback and reduced models of linear systems. Among the remarkable mathematical results associated with the theory are the Kalman-Yakubovich-Popov lemma describing the relation between frequency domain inequalities and stabilizing solutions of Riccati equations, the Adamyan-Arov-Krein theorem on Hankel optimal model reduction, and the state space solution of H-Infinity optimization problem. Practical implementation of these basic concepts and mathematical theory rely on numerical linear algebra and convex optimization engines: linear equation solvers, Schur decomposition, and LMI optimization (semidefinite programming).

1.1 Example: Inverted Pendulum with Control Delay

Consider the task of designing a linear controller K (more specifically, a finite order LTI system with rational transfer function $K(s)$) with input $\theta = \theta(t)$ (angular position of a pendulum) and output $v = v(t)$ (actuator torque command) to stabilize $\theta = \theta(t)$ at the upright position $\theta = 0$ (see Figure 1.1), while assuming that pendulum equations are given, after appropriate normalization of v, by

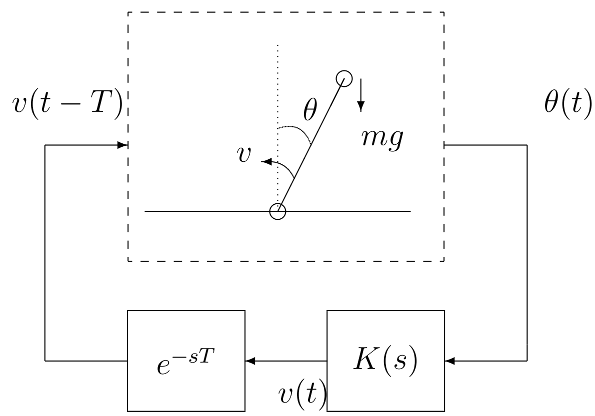

Figure 1.1: Control of Inverted Pendulum

$$\ddot{\theta}(t) = \omega_0^2 \sin(\theta(t)) + v(t - T), \tag{1.1}$$

which means a constant control delay by T units of time and absence of friction. Given the values of T and $\omega_0^2 = g/L$ (where g is free fall acceleration and L is pendulum length), a basic task of feedback design is to assure local stability of the equilibrium $\theta(t) \equiv 0$.

In this example, the original model is "complex", in the sense that it involves a nonlinear component $\theta(t) \mapsto \sin(\theta(t))$ and an infinite order LTI component $v(t) \mapsto v(t-T)$. Application of modern control principles to the stabilization problem begins with finding good finite order LTI approximation of (1.1), complete with error bounds, expressed in terms of *L2 gains* or *Integral Quadratic Constraints*.

Memoryless nonlinear transformations, such as $\theta(t) \mapsto \sin(\theta(t))$, cannot be approximated by the linear ones with arbitrary accuracy. As a compromise, higher quality of approximation in a certain region of values of the input is achieved at the expense of lowering the quality in the complementing region. If the range of admissible values of $\theta(t)$ is $[-\theta_0, \theta_0]$, where $\theta_0 \in (0, \pi)$, one can use $\sin(\theta(t)) \approx k\theta(t)$, where

$$k = \frac{1}{2}\left(1 + \frac{\sin(\theta_0)}{\theta_0}\right)$$

defines the center line $\{(\theta, k\theta) : \theta \in \mathbf{R}\}$ of the (non-convex) cone spanned by the points $(\theta, \sin(\theta))$ with $|\theta| \le \theta_0$. This leads to a finite order LTI representation

$$\sin(\theta(t)) = k_{\text{sin}}\theta(t) + d_{\text{sin}}^{-1} w_{\text{sin}}(t), \tag{1.2}$$

visualized by the block diagram of Figure 1.2, where w_{sin} is a scaled *modeling error signal*,

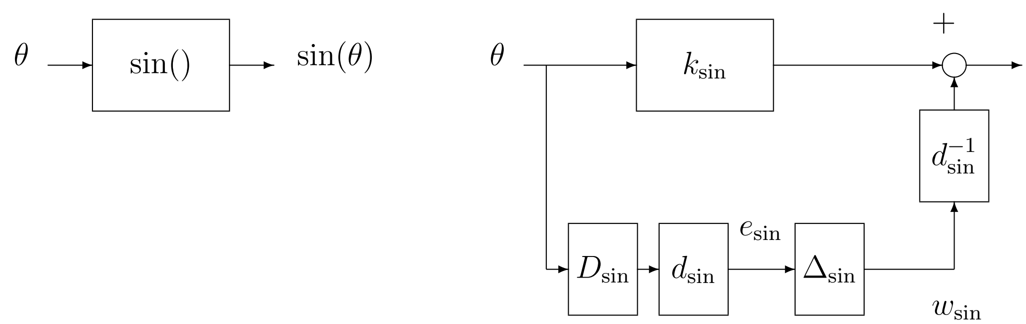

Figure 1.2: Linearization of $\sin(\cdot)$

Δ_{sin} is the *modeling error system*, mapping e_{sin} to w_{sin} according to

$$w_{\text{sin}} = d_{\text{sin}}(\sin(\theta) - k_{\text{sin}}\theta), \tag{1.3}$$

and

$$e_{\text{sin}}(t) = d_{\text{sin}} D_{\text{sin}} \theta(t), \tag{1.4}$$
$$D_{\text{sin}} = 0.5(1 - \sin(\theta_0)/\theta_0).$$

A simple calculation shows that

$$\int_0^T |w_{\text{sin}}(t)|^2 dt \le \int_0^T |e_{\text{sin}}(t)|^2 dt, \tag{1.5}$$

provided that $|\theta(t)| \le \theta_0$ for all t. Condition (1.5) serves as an *L2 gain bound* for the modeling error system Δ_{sin}, quantifying the approximation error subject to the assumption $|\theta(t)| \le \theta_0$. Approximation of nonlinear subsystems by the linear ones, combined with the use of conditional energy gain bounds in stability and performance analysis, is an important component of modern control.

For the delay transformation $v(t) \mapsto v(t-T)$, one can use representation of the form

$$\begin{aligned} v(t-T) &= \hat{v}(t) + d_{del}^{-1} w_{del}(t), && (1.6)\\ \hat{v}(t) &= C_d x_d(t) + D_d v(t), && (1.7)\\ \dot{x}_d(t) &= A_d x_d(t) + B_d v(t), && (1.8) \end{aligned}$$

where A_d, B_d, C_d, D_d are coefficient matrices of a stable LTI state space model approximating the delay, $w_{del}(t)$ is scaled delay modeling error, and $d_{del} > 0$ is a scaling parameter. A pure delay cannot be approximated by a finite order LTI system with arbitrary accuracy, because the error is always large for high frequencies. As a compromise, the approximation error is quantified by establishing an energy bound of the form

$$\int_0^T |w_{del}(t)|^2 dt \le \int_0^T |e_{del}(t)|^2 dt, \qquad (1.9)$$

where

$$e_{del}(t) = d_{del}(C_e x_d(t) + D_e v(t)) \qquad (1.10)$$

is an auxiliary output of (1.8), defining e_{del} as the result of applying a low-pass filter to v. For example, when $T > 0$ is small enough, which makes the delay easier to approximate, the coefficients in (1.7),(1.8),(1.10) can be defined by

$$D_d + C_d(sI - A_d)^{-1} B_d = \frac{1 - Ts/2}{1 + Ts/2}, \quad D_e + C_e(sI - A)^{-1} B_d = \rho \frac{s + a}{1 + Ts/2},$$

where

$$\rho \ge \max_{\omega \in \mathbf{R}} \left| \frac{1 + j\omega T}{j\omega + a} \left(e^{-j\omega T} - \frac{1 - j\omega T/2}{1 + j\omega T/2} \right) \right|,$$

and $a > 0$ is a parameter. The theory of efficient approximation of high (or infinite) order LTI systems by systems of low order (*model reduction*) is an important part of modern control.

Combining equations (1.1) - (1.10) with state space controller equations

$$\begin{aligned} v(t) &= C_f x_f(t) + D_f \theta(t), && (1.11)\\ \dot{x}_f(t) &= A_f x_f(t) + B_f \theta(t), && (1.12) \end{aligned}$$

where A_f, B_f, C_f, D_f are constant real coefficient matrices, and $x_f = x_f(t)$ is the *state* of the controller, yields LTI model (1.11)-(1.16) (see also Figure 1.3),

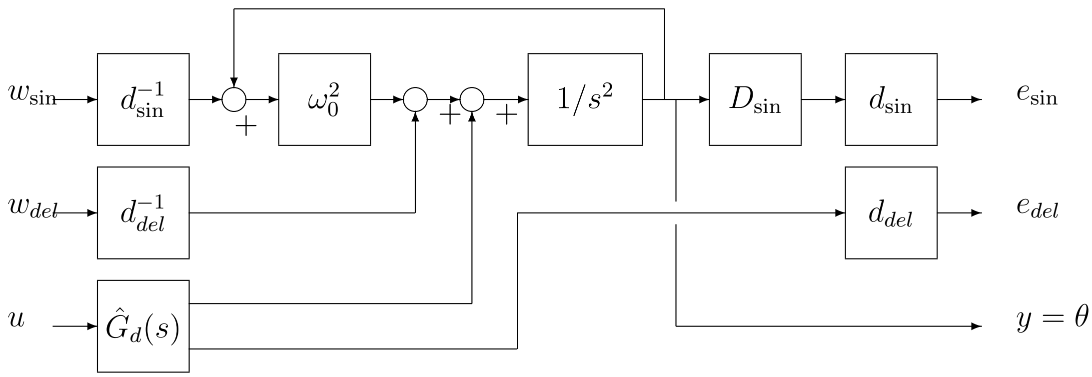

Figure 1.3: Linearized pendulum model

$$\ddot{\theta} - \omega_0^2\theta = \omega_0^2 d_{\sin}^{-1} w_{\sin} + C_d x_d(t) + D_d v(t) + d_{del}^{-1} w_{del}, \quad (1.13)$$
$$\dot{x}_d = A_d x_d + B_d v, \quad (1.14)$$
$$e_{\sin} = d_1 D_{\sin}\theta, \quad (1.15)$$
$$e_{del} = d_{del}(C_e x_d + D_e v, \quad (1.16)$$

where L2 gain from $e = [e_{\sin}; e_{del}]$ to $w = [w_{\sin}; w_{del}]$ is known to be not larger than 1. According to the *small gain theorem*, a controller (1.11), (1.12) stabilizes system (1.1) if the L2 gain from w to e in the LTI model (1.11)-(1.16) is less than 1. The feedback design is now reduced to finding the coefficients of controller (1.11), (1.12) and the positive coefficients $d_{\sin}, d_{del}$ in (1.13) - (1.16) minimizing the L2 gain from w to e.

For $d_{\sin}, d_{del} > 0$ fixed, one can use the technique of designing LTI feedback control minimizing an L2 gain in an LTI closed loop system, a major component of modern control called *H-Infinity optimization*. Similarly, for a fixed controller (1.11), (1.12), one can use *semidefinite programming* to minimize the L2 gain as a function of scaling parameters $d_{\sin}, d_{del}$. Combined together, H-Infinity optimization and semidefinite programming form an ad-hoc procedure of *D-K iteration* for designing robust feedback controllers.

Once a controller (1.11), (1.12) stabilizing (1.1) subject to the assumption $|\theta(t)| \le \theta_0$ is designed, it remains to calculate the range of initial conditions in (1.1), (1.11), (1.12) which guarantee that the assumption $|\theta(t)| \le \theta_0$ is satisfied for all $t \ge 0$. This can be achieved by extracting Lyapunov functions for the closed loop system from the information contained in $d_{\sin}$ and d_{del}, made possible by the theory of system analysis using *Integral Quadratic Constraints*.

1.2 Example: Active Damping for a Flexible Structure

Consider a situation where two low quality sensors and two weak actuators are used to reduce oscilations induced in an undamped flexible medium by noisy disturbances. For simplicity, the flexible medium is modeled as a chain of $4m$ (where m can be very large) identical point masses moving in a single dimension and connected by identical springs, with the sensors and actuators located at one quarter chain length from both ends of the chain, as shown on Figure 1.4.

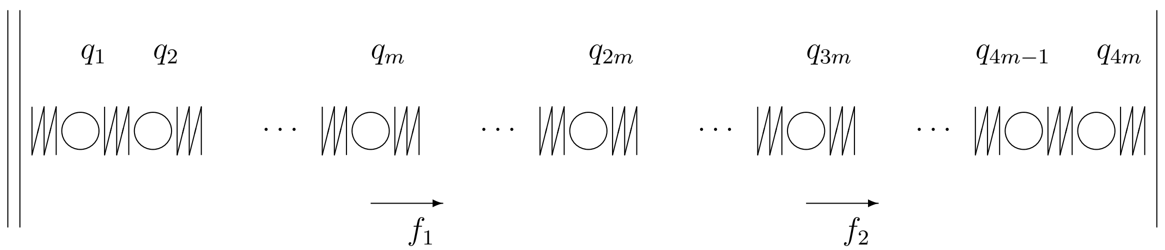

Figure 1.4: spring chain

System equations are assumed to have the form

$$\begin{aligned}
\ddot{q}_1 &= r(q_2 - q_1) + bw_1, \\
\ddot{q}_k &= r(q_{k-1} + q_{k+1} - 2q_k) + bw_k, \quad (k = 2, \dots, m-1) \\
\ddot{q}_m &= r(q_{m-1} + q_{m+1} - 2q_m) + bw_m + f_1, \\
\ddot{q}_k &= r(q_{k-1} + q_{k+1} - 2q_k) + bw_k, \quad (k = m+1, \dots, 3m-1) \\
\ddot{q}_{3m} &= r(q_{3m-1} + q_{3m+1} - 2q_{3m}) + bw_{3m} + f_2, \\
\ddot{q}_k &= r(q_{k-1} + q_{k+1} - 2q_k) + bw_k, \quad (k = 3m+1, \dots, 4m-1) \\
\ddot{q}_{4m} &= r(q_{4m-1} - q_{4m}) + bw_{4m},
\end{aligned}$$

where q_k is the displacement of the k-th mass, f_1, f_2 are the actuator forces, w_k is the noisy force disturbing the k-th mass, and r, b are constant positive coefficients. In addition, the measurement process is modeled by

$$\begin{aligned}
y_1 &= g_1 + v_1, \\
y_2 &= g_2 + v_2, \\
\dot{g}_1 &= a(q_m - g_1), \\
\dot{g}_2 &= a(q_{3m} - g_2),
\end{aligned}$$

where y_1, y_2 are the mass position measurements, the differential equations for g_1, g_2 represent sensor inertia ($a > 0$ is a time constant parameter), and v_1, v_2 are measurement noises. The objective is to design a feedback law defining $f = [f_1; f_2]$ as a causal finite order LTI function of measurement $y = [y_1; y_2]$ which minimizes the mean square value of $|f|^2 + |q_{2m}|^2$ while w_i, v_i are modeled as independent white noise stochastic processes.

The practical difficulty of this feedback design setup (apart from combining multiple sensors and multiple actuators) is due to the very high order of the exact model. While H2 optimization is available, in theory, to find the optimal feedback law $y(\cdot) \mapsto f(\cdot)$, the resulting optimal controller has the same order as the open loop system (inputs f, w, v, outputs y, q_{2m}), which is impractical for large m. Therefore, for $m \gg 1$, a model reduction technique is to be applied before feedback optimization, to produce a reduced order approximated description of the open loop map. To account for the approximation error, the resulting model will contain an uncertainty block, and will be solved by combining H-Infinity optimization with semidefinite programming.