Chapter 2

Signals and Systems Conventions

This chapter introduces elementary terminology of systems and signals. While pretty common and standard, the framework is treated slightly differently by different literature sources, hence the need to introduce a clear set of consistent conventions to be used in this class.

2.1 Signals

Signals are intended to represent time-stamped data. In particular, a *discrete time (DT) signal* w of dimension m is a semi-infinite sequence $w = (w(0), w(1), w(2), \dots)$ of vectors $w(t) \in \mathbb{R}^m$, or, equivalently, a function $w : \mathbb{Z}_+ \to \mathbb{R}$, where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ denotes the set of non-negative integers. We will use the notation ℓ^m (or simply ℓ for $m = 1$) to denote the set of all m-dimensional DT signals.

The situation with *continuous time* (CT) is more complicated. An m-dimensional CT signal w is a function $w :\ \mathbb{R}_+ \to \mathbb{R}^m$ (where $\mathbb{R}_+$ denotes the real half-axis $\mathbb{R}_+ = [0, \infty)$) which satisfies additional conditions of *regularity*, essentially making sure that some integrals associated with the signal are well defined. In this class, the set $\mathcal{L}^m$ (or simply $\mathcal{L}$ for $m = 1$) of all m-dimensional CT signals consists of all *measureable*[1] functions $w :\ [0, \infty) \to \mathbb{R}^m$, such that

$$\int_0^T |w(t)|^2 dt < \infty \quad \forall\, T > 0, \tag{2.1}$$

where, for a vector $a \in \mathbb{R}^m$, the expression $|a|$ denotes its Euclidean norm.

[1] See a textbook on real analysis for details!

Example 2.1. The function $w_r : [0, \infty) \to \mathbb{R}$ defined by $w_r(t) = t^r$, where $r \in \mathbb{R}$ is a parameter, is a CT signal if and only if $r > -0.5$

The need to have a specification for what constitutes a "*signal*" is akin to the need to have a right definition of a *number*. For example, if "number" means "rational number" (which was likely the case for ancient mathematicians and engineers), one cannot think accurately about the length of diagonal in a square of unit side length; if "number" means "algebraic number" (a root of a polynomial with integer coefficients), then one cannot think accurately about the length of half-arc in a circle with unit radius length, etc.: the set of available numbers should be rich enough to include all situations of possible interest.

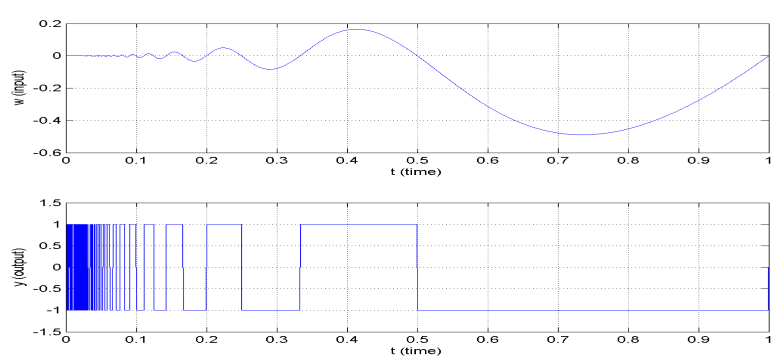
The same applies to defining what constitutes a "signal": some restrictions, seemingly common sense, may eventually become inconveniences. For example, while signals in a physical system are typically nicely bounded, it is inconvenient to limit models to bounded signals only, as the possibility of unbounded internal signals in simplified models is commonly used to prove instability of feedback interconnections. In a similar fashion, while convenient in many applications, is it not necessarily a good idea to restrict CT signals to be in the class of *piecewise-continuous* functions $w : [0, \infty) \mapsto \infty$, i.e. those for which there exists a sequence $0 = \tau_0 < \tau_1 < \tau_2 < \dots$ of real numbers such that $\lim t_k = \infty$ as $k \to \infty$, and w is continuous a every point $t \in [0, \infty)$ except, possibly, $t = \tau_k$, and the limits $\lim_{t \to \tau_+}$, $\lim_{t \to \tau_-}$ do exist. Indeed, letting a nice continuous signal, such as

$$w(t) = \begin{cases} t^2 \sin(t), & t > 0, \\ 0, & t = 0, \end{cases}$$

pass through a reasonable "switch" transformation, such as

$$w(\cdot) \;\mapsto\; y(\cdot): \quad y(t) = \begin{cases} 0, & w(t) \le 0, \\ 0, & w(t) > 0, \end{cases}$$

will result in a function $y(\cdot)$ which is *not* piecewise continuous (see the figure below).



With DT signals, it is possible to view *all* possible functions of time as signals (this is what the definition of ℓ^m does). The situation with continuous time is more complicated: it is not possible to view arbitrary functions $w:\ (0,\infty)\to\mathbb{R}^m$ as signals, as most of such functions do not allow integration in any form.

2.2 Systems

In this class, a *DT system* S with m-dimensional input, k-dimensional output, and *boundary conditions set* X is a function S mapping signal/boundary condition pairs $(w,x_0)\in\ell^m\times X$ to non-empty subsets $S(w,x_0)$ of ℓ^k. In other words, a system is defined when, for arbitrary *initial condition data* $x_0\in X$ and *input signal* $w\in\ell^m$, the set $S(w,x_0)=\{y\}$ of *all possible output signals* $y\in\ell^k$ is specified. The set of all such systems will be denoted as $\mathcal{S}^{m,k}_{DT}(X)$.

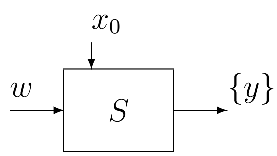


Figure 2.1: A general system

Similarly, a *CT system* S with m-dimensional input, k-dimensional output, and *boundary conditions set* X is a function S mapping pairs (w,x_0) from $\mathcal{L}^m\times X$ to non-empty subsets $S(w,x_0)$ of $\mathcal{L}^k$. The set of all such systems will be denoted as $\mathcal{S}^{m,k}_{CT}(X)$.

These definitions are complicated enough to require an (apologetic) explanation.

Consider, for example, the "leaking" DT integrator relation

$$y(t)=0.5(y(t-1)+w(t))\quad (t=1,2,3,\dots), \tag{2.2}$$

where $w\in\ell$ is the input and $y\in\ell$ is the output. One can begin by assuming that $y(0)=0$ (i.e. the output signal was reset to zero at $t=0$), which makes the output

$$y(t)=\sum_{\tau=0}^{t}2^{-\tau-1}w(t-\tau)$$

uniquely defined by the input. This is compatible with the common and convenient, if simplistic, view of a DT system as that of a function $F:\ \ell^m\to\ell^k$ mapping signals to signals: $y=F(w)$ means that signal $y\in\ell^k$ is the response of system F to input $w\in\ell^m$.

A functional model assumes that system response is uniquely defined by the input, which is rarely the case: variations in *boundary conditions* (or, more specifically, in the case of *causal* sysytems, variations in *initial conditions*) lead to variations in system response to a given input signal. For example, it is not always reasonable to assume that the initial value $y(0)$ in (2.2) has any particular value. To account for the non-uniqueness of system response to a given input, one can consider a *behavioral model* B, which is simply the set $B = \{(w, y)\}$ of *all* possible input-output pairs (in the example under consideration, the set of all pairs (w, y) satisfying (2.2)). This is adequate for most instances of *worst-case analysis* of systems, which is concerned with proving that some conditions are satisfied for *all* possible combinations of signals within a system. For example, the (worst case) peak-to-peak gain of behavioral model $B = \{(w, y)\}$ will be defined as the maximal lower bound of $\gamma > 0$ such that for every $(w, y) \in B$

$$\sup_T \{\sup_t |y(t)| - \gamma \sup_t |w(t)|\} < \infty$$

(a standard L2 gain calculation algorithm, to be discussed in later chapters, shows that the behavioral model B defined by (2.2) has peak-to-peak gain of 1).

In contrast, behavioral modeling is poorly compatible with *probabilistic* analysis, which typically assumes that the set of all possible inputs is parametrized by a hidden "case" parameter $\theta \in [0, 1]$, and uses expected values (i.e. integrals over θ) to quantify system behavior. For example, the commonly used "white nose to second moment" gain of a DT model is typically defined as the minimal upper bound of

$$\lim_{t \to \infty} \mathbf{E}[|y(t)|^2],$$

subject to w being a sequence of independent zero mean normalized Gaussian random variables. The problem with behavioral modeling is that, with no provision for "fixing" the initial condition of an input-output pair $(w, y) \in B$, there is no way of bounding the second moment $\mathbf{E}[|y(t)|^2]$, at least not for the behavioral model derived from (2.2).

This suggests the introduction of a *boundary condition* parameter x_0, ranging over some fixed set X, so that output y becomes a function $y = I(w, x_0)$ of the pair (w, x_0). This makes it possible to fix the initial condition x_0 when considering system response to a parameterized family of inputs. In particular, the "white nose to second moment" gain can be defined by

$$\sup_{x_0} \lim_{t \to \infty} \mathbf{E}[|y(t)|^2], \quad \text{where} \quad y = I(w, x_0). \tag{2.3}$$

In the case of the system defined by (2.2), it is natural to let $X = \mathbb{R}$ and assume the relation $y(0) = x_0$, which leads to $y = I(w, x_0)$ expressed as

$$y(t) = 2^{-t}x_0 + \sum_{\tau=0}^{t} 2^{-\tau-1}w(t-\tau),$$

with the "white nose to second moment" gain easily computed at 1/3.

The functional models of systems with boundary conditions are general enough to cover most of the possible applications. However, not all non-uniqueness of system response to a given input is due to variance in boundary conditions.

Consider the following model of evolution of velocity $v = v(t)$ of a single degree of freedom unit mass object subject to external driving force input w and dry friction force u:

$$\dot{v}(t) = w(t) + u(t), \quad u(t) \begin{cases} \in [-1, 1], & v(t) = 0, \\ = -1 + \sqrt{v(t)} - v(t), & v(t) > 0, \\ = 1 - \sqrt{v(t)} + v(t), & v(t) < 0. \end{cases} \tag{2.4}$$

It is natural to suggest $X = \mathbb{R}$ and $x_0 = v(0)$, for a boundary conditions functional model $v = I(w, x_0)$. However, this approach does not quite work in this case. Indeed, as can be shown by solving the resulting ordinary differential equation explicitly, with $x_0 = 0$ and $w(t) \equiv 1$, system equations (2.4) have an uncountable set of possible solutions $v = v(t)$. This observation has the following interpretation:

- with the mass initially at rest ($v(0) = 0$), the value $w_0 = 1$ is critical for the driving force: with $|w(t)| < 1$, the velocity will remain zero; with $w(t) > 1$, the mass will begin moving;

- with the the driving force *exactly* at the critical value $w(t) \equiv 1$, there are, informally speaking, two possible outcomes: either the mass remains at rest forever (i.e. $v(t) = 0$ for all $t > 0$), or it starts moving at some time $t_* > 0$, which means $v(t) = 0$ for $t \leq t_*$, and $v(t) > 0$ for $t > t_*$.

The example shows that, in some models of potential interest, complete specification of boundary conditions is still not enough to make output uniquely defined by the input. This is exactly the reason for using a set-valued function format $y \in S(w, x_0)$ in the definition of the system model class in the beginning of this section, instead of the more straightforward signal-valued function format $y = S(w, x_0)$.

2.3 Causality, Linearity, and Time Invariance

The standard system qualities, causality, linearity, and time invariance, can be defined for system models, introduced in the previous section.

2.3.1 Causality

Let P_T denote the *past projection* operation. For a DT signal $w \in \ell^m$ and $T \geq 0$, signal $v = P_T w \in \ell^m$ is defined by

$$v(t) = \begin{cases} w(t), & t \leq T, \\ 0, & t > T. \end{cases}$$

The definition is exactly the same in the CT case, except that $v = P_T w \in \mathcal{L}^m$ is a CT signal when $w \in \mathcal{L}^m$. As usually, for a set of signals W,

$$P_T W \stackrel{\text{def}}{=} \{P_T w : \; w \in W\}.$$

A DT system $S \in \mathcal{S}_{DT}^{m,k}(X)$ is called *causal* if $P_T S(w, x_0) = P_T S(\tilde{w}, x_0)$ for every two signals $w, \tilde{w} \in \ell^m$ such that $P_T w = P_T \tilde{w}$. The definition is the same in the CT case (except that $S \in \mathcal{S}_{CT}^{m,k}(X)$, and $w, \tilde{w} \in \mathcal{L}^m$), and means that the set of the possible output pasts may depend on the boundary condition and/or the past of the input, but not on the future of the input.

It is typically very straightforward to determine causality (or lack thereof) of a given system model.

Example 2.2. Consider three *different* models "related" to the difference equation

$$y(t+1) = 2y(t) + \text{sat}(w(t)) \quad (t \in \mathbb{Z}_+), \tag{2.5}$$

with $w \in \ell$ being the input, and $y \in \ell$ being the output, where $\text{sat}() : \; \mathbb{R} \mapsto \mathbb{R}$ is the "ideal limiter" function defined by

$$\text{sat}(a) = \begin{cases} a, & a \in [-1, 1], \\ 1, & a > 1, \\ -1, & a < -1. \end{cases}$$

The first model is $S_0 \in \mathcal{S}_{DT}^{1,1}(\{0\})$ (where the "$\{0\}$" argument means that no boundary conditions are to be specified) is given by

$$S_0(w, x_0) = \{y : \quad y(t) = \sum_{\tau=1}^{t} 2^{\tau-1} \text{sat}(w(t-\tau)) \; \forall \; t \in \mathbb{Z}_+\}.$$

This corresponds to setting $y(0) = 0$ in (2.5).

The second model is $S_1 \in \mathcal{S}^{1,1}_{DT}(\mathbb{R})$ (i.e. has boundary conditions parametrized by real numbers), and is given by

$$S_1(w, x_0) = \{y : \quad y(t) = 2^t x_0 + \sum_{\tau=1}^{t} 2^{\tau-1}\mathrm{sat}(w(t-\tau)) \ \forall\ t \in \mathbb{Z}_+\}.$$

This corresponds to selecting $y(0) = x_0$ as the boundary condition in (2.5).

The third model is $S_2 \in \mathcal{S}^{1,1}_{DT}(0)$, and is given by

$$S_2(w, x_0) = \{y : \quad y(t) = \sum_{\tau=0}^{\infty}(-2)^{-\tau-1}\mathrm{sat}(w(t+\tau)) \ \forall\ t \in \mathbb{Z}_+\}.$$

This corresponds to selecting

$$\lim_{t\to\infty} y(t) = 0$$

as the boundary condition in (2.5).

While the input-output pairs in S_0, S_1 and S_2 satisfy equations (2.5), the resulting systems are very different.

The models S_0, S_1 are *causal*. The *proof* of this goes by observing that $S_0(w, x_0), S_1(w, x_0)$ are single-element sets, with the only element being the signal $y \in \ell$ defined in such a way that, for every $t \in \mathbb{Z}_+$, $y(t)$ is uniquely determined by $x_0, w(0), w(1), \dots, w(t)$.

The model S_2 is *not causal*. As usually, the *lack* of some property is proven by an example. Consider $w_0, \tilde{w}_0 \in \ell$ defined by

$$w_0(t) \equiv 0, \quad \tilde{w}_0(t) = \begin{cases} 1, & t = 1, \\ 0, & t \neq 1. \end{cases}$$

Then, for $T = 0$, we have

$$P_T w_0 = P_T \tilde{w}_0 = \{w_0\},$$

while the $P_T S_2(w_0, 0) = \{w_0\}$, and $P_T S_2(\tilde{w}_0, 0) = \{\tilde{y}_0\}$ is the set with a single element y_0, defined by

$$\tilde{y}_0(t) = \begin{cases} 1, & t = 0, \\ 0, & t > 0. \end{cases}$$

Hence $P_T S_2(w_0, 0) \neq P_T S_2(\tilde{w}_0, 0)$ despite $P_T w_0 = P_T \tilde{w}_0$ which completes the formal proof of the lack of causality.

2.4 Linearity

A DT system $S \in \mathcal{S}_{DT}^{m,k}(X)$ is called *linear* when X is a *vector space* (i.e. when it has a special "zero" element $0 \in X$ and operations of "addition" $+ : \ X \times X \to X$ and "multiplication" $\cdot : \ \mathbb{R} \times X \to X$ defined, satisfying all the usual axioms), such that, for all $w_1, w_2 \in \ell^m$, $x_1, x_2 \in X$, and $c_1, c_2 \in \mathbb{R}$, signal y belongs to $S(c_1 w_1 + c_2 w_2, c_1 x_1 + c_2 x_2)$ if and only if there exist $y_1 \in S(w_1, x_1)$ and $y_2 \in S(w_2, x_2)$ such that $y = c_1 y_1 + c_2 y_2$.

This is a standard definition of linearity as "validity of the superposition law", and is the same in the CT case, with the obvious replacement of ℓ^m by $\mathcal{L}^m$.

2.5 Time Invariance

Let F_T denote the "*future of the signal*" operation. For a DT signal $w \in \ell^m$ and $T \in \mathbb{Z}_+$, signal $v = F_T w \in \ell^m$ is defined by $v(t) = w(t+T)$. The definition is exactly the same in the CT case, except that, for $w \in \mathcal{L}^m$, possible values of T range over $\mathbb{R}_+$, and $v = F_T w \in \mathcal{L}^m$ is a CT signal. As usually, for a set of signals W,

$$F_T W \stackrel{\text{def}}{=} \{F_T w : \ w \in W\}.$$

A DT system $S \in \mathcal{S}_{DT}^{m,k}(X)$ is called *time invariant* when there exists a function f mapping pairs $(w, x_0) \in \ell^m \times X$ to subsets $f(w, x_0) \subset \mathbb{R}^k \times X$, in such a way that $y \in S(w, x_0)$ if and only if there exists a pair $(y_0, x_1) \in f(w, x_0)$ such that $y(0) = y_0$ and $F_1 y \in S(F_1 w, x_1)$.

Informally speaking, the definition establishes a "recursive" model

$$(x_{t+1}, y(t)) \in f(w_t, x_t), \quad w_{t+1} = F_1 w_t, \quad w_0 = w$$

for producing all possible system responses.

A CT system $S \in \mathcal{S}_{CT}^{m,k}(X)$ is called *time invariant* when for every $T > 0$ there exists a function f_T mapping pairs $(w, x_0) \in \mathcal{L}^m \times X$ to subsets $f(w, x_0) \subset P_T \mathcal{L}^k \times X$, in such a way that $y \in S(w, x_0)$ if and only if there exists a pair $(y_T, x_T) \in f_T(w, x_0)$ such that $P_T y = y_T$ and $F_T y \in S(F_T w, x_T)$.

In the case of systems S_0, S_1, S_2 from Example 2.2, models S_1 and S_2 are time invariant, while S_0 is not.

To *prove* that S_0 is not time invariant, note that, since the set of possible values of $y(1)$ is described by the same function f which decides the set of possible values of $y(0)$, in the sense that

$$(y(0), x_1) \in f(w, x_0), \quad (y(1), x_2) \in f(F_1 w, x_1),$$

the range of possible values of $y(0)$ must contain the range of possible values of $y(1)$. For system S_0, the latter is $\mathbb{R}$ while the former is $\{0\}$, which proves the absence of time invariance.

To *prove* that S_1 is time invariant, define f as the function mapping $(w, x_0) \in \ell \times \mathbb{R}$ to the set consisting of the single element (y_0, x_1), where

$$y_0 = x_0, \quad x_1 = 2x_0 + \mathrm{sat}(w(0)),$$

and verify that is satisfies the required conditions.

To *prove* that S_2 is time invariant, define f as the function mapping $(w, 0) \in \ell \times \{0\}$ to the set consisting of the single element $(y_0, 0)$, where

$$y_0 = \sum_{\tau=0}^{\infty} (-2)^{-\tau-1} \mathrm{sat}(w(\tau))$$

and verify that is satisfies the required conditions.

The informal explanation is that model S_0 has turned out to be time-varying because it favors the specific time instance $t = 0$ to set up the initial conditions.

2.6 System Gains and Stability

The need to measure the degree of sensitivity of system response to its input surfaces in may applications: it can quantify the degree of disturbance rejection in feedback control, or the accuracy of approximating one model by another, or quality of a signal processing algorithm.

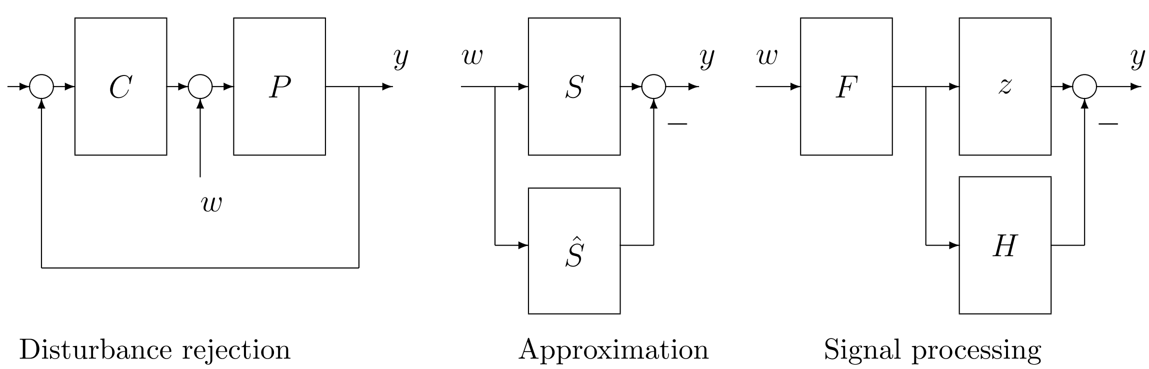


Figure 2.2: Use of system gains from w to y

One has to realize that, depending on the objectives of feedback system design and analysis, there are many significantly different notions of system gains.

2.6.1 Finite L2 Gain Stability

Comparing "energy" (understood as integral of signal squared) of input and output is a common approach to quantifying sensitivity. Even within this framework, there are several distict versions of system gain, differentiated by the way they treat initial conditions.

Asymptotic L2 Gain

The asymptotic *L2 gain* (or simply *L2 gain*) of a DT system $S \in \mathcal{S}_{DT}^{m,k}(X)$ is the maximal lower bound of $\gamma \geq 0$ such that for every $w \in \ell^m$, $x_0 \in X$, and $y \in S(w, x_0)$ there exists a constant $c \in \mathbb{R}$ such that

$$\sum_{t=0}^{T} |y(t)|^2 \leq c + \gamma^2 \sum_{t=0}^{T} |w(t)|^2. \tag{2.6}$$

The informal meaning behind this definition is for the "output energy" (i.e. the sum from the left side of inequality (2.6)) to be bounded by the "initial energy" accumulated within the system (constant c from (2.6)) plus a coeffieint (square of the L2 gain) times the "input energy" (the sum from the right side of (2.6)).

Asymptotic L2 gain (or simply *L2 gain*) of a CT system $S \in \mathcal{S}_{CT}^{m,k}(X)$ is defined in a similar way, with ℓ^m replaced by $\mathcal{L}^m$, and (2.6) replaced by

$$\int_0^T |y(t)|^2 dt \leq c + \gamma^2 \int_0^T |w(t)|^2 dt. \tag{2.7}$$

L2 gain is well-defined (though can be hard to calculate) for every system, and is either a non-negative number or $+\infty$. A system with a finite L2 gain is called *L2 gain stable*.

Fixed State L2 Gain

The fixed state *L2 gain* of a DT system $S \in \mathcal{S}_{DT}^{m,k}(X)$ is the maximal lower bound of $\gamma \geq 0$ such that for every $x_0 \in X$ there exists a constant $c \in \mathbb{R}$ such that the inequality (2.6) holds for every $w \in \ell^m$ and $y \in S(w, x_0)$. Compared to the definition of the *asymptotic* L2 gain, the constant c in (2.6) is not allowed to depend on w or y: it must be a function of the boundary condition x_0.

Fixed state L2 gain (or simply *L2 gain*) of a CT system $S \in \mathcal{S}_{CT}^{m,k}(X)$ is defined in a similar way, as the maximal lower bound of $\gamma \geq 0$ such that for every $x_0 \in X$ there exists a constant $c \in \mathbb{R}$ such that the inequality (2.7) holds for every $w \in \mathcal{L}^m$ and $y \in S(w, x_0)$.

L2 Induced Norm

In applications which assume that initial accumulated energy of every subsystem is zero, the slightly simpler notion of *L2 induced norm* may be suitable.

The *L2 induced norm* of a DT system $S \in \mathcal{S}^{m,k}_{DT}(X)$ is the maximal lower bound of $\gamma \geq 0$ such that

$$\sum_{t=0}^{\infty} |y(t)|^2 \leq \gamma^2 \sum_{t=0}^{\infty} |w(t)|^2 \tag{2.8}$$

for every $x_0 \in X$, $w \in \ell^m$, and $y \in S(w, x_0)$, such that the sum on the right side of (2.8) is finite.

The definition of L2 induced norm for CT a system $S \in \mathcal{S}^{m,k}_{CT}(X)$ is similar, with ℓ^m replaced by $\mathcal{L}^m$, and the sums in (2.8) replaced by the corresponding integrals.

L2 Stability

L2 gains and induced norms are well-defined (though can be hard to calculate) for every system, and are either non-negative numbers or $+\infty$. A system with a finite L2 gain is called *L2 gain stable* (with similar definitions of *fixed gain L2 gain stability*, and *induced L2 norm stability*.

Example 2.3. DT system $S \in \mathcal{S}^{1,1}_{DT}(\{0\})$ defined by

$$S(w, 0) = \{y : \quad y(t) = \sin(w(t)) \ \forall \ t \in \mathbb{Z}_+\}$$

has all three L2 sensitivity measures (L2 gain, fixed state L2 gain, L2 induced norm) equal to 1, and is therefore stable according to all three definitions.

For example, to see that L2 gain of S is *not larger* than 1, use the inequality

$$|\sin(a)| \leq |a| \quad \forall \ a \in \mathbb{R},$$

which establishes (2.6) with $c = 0$ for $\gamma \geq 1$.

To see that L2 gain of S is *not smaller* than 1, use the input-output pair $w(t) \equiv \delta$, $y(t) \equiv \sin(\delta)$, where $\delta > 0$ is a small constant. For this input-output pair, condition (2.6) means

$$T \sin(\delta)^2 \leq c + \gamma^2 T \delta^2 \quad \forall \ T \geq 0,$$

which implies

$$\gamma^2 \geq \frac{\sin^2(\delta)}{\delta^2} \quad \forall \ \delta > 0.$$

Letting $\delta \to 0$ yields $\gamma^2 \geq 1$, hence $\gamma \geq 1$.

Example 2.4. System $S \in \mathcal{S}^{1,1}_{CT}(\{0\})$ defined by

$$S(w,0) = \{y : \quad y(t) = e^{-t}(1 + w(t)) \; \forall \; t \in \mathbb{Z}_+\}$$

has (asymptotic) L2 gain of 0, fixed state L2 gain of 1, and induced L2 norm of ∞.

2.6.2 Finite Peak-to-Peak Gain Stability

If maximal amplitude of a signal is more of a concern than its energy, the following definition could be more appropriate.

Define the *peak-to-peak gain* of a DT system $S \in \mathcal{S}^{m,k}_{DT}(X)$ as the maximal lower bound of $\gamma \geq 0$ such that for every $w \in \ell^m$, $x_0 \in X$, and $y \in S(w, x_0)$ there exists a constant $c \in \mathbb{R}$ such that

$$\max_{t \leq T} |y(t)| \leq c + \gamma \max_{t \leq T} |w(t)|. \tag{2.9}$$

The informal meaning behind this definition is for the "peak amplitude" of the output (i.e. the maximum from the left side of inequality (2.9)) to be bounded by a constant depending on "initial conditions" (constant c from (2.9)) plus a coeffieint (the peak-to-peak gain) times the "peak amplitude" of the input (the maximum from the right side of (2.9)).

Peak-to-peak gain of a CT system $S \in \mathcal{S}^{m,k}_{CT}(X)$ is defined in a similar way, with ℓ^m replaced by $\mathcal{L}^m$.

Peak-to-peak gain is well-defined (though can be hard to calculate) for every system, and is either a non-negative number or $+\infty$. A system with a finite peak-to-peak gain is called *peak-to-peak gain stable*.

Note that the system from Example 2.3 has peak-to-peak gain of zero. (Indeed, since $|\sin(a)| \leq 1$ for all $a \in \mathbb{R}$, condition (2.9) is satisfied with $c = 1$ for *all* $\gamma \geq 0$.)

2.6.3 Incremental L2 Stability

The "gain" stability definitions from the previous section are reasonable in situations when stability means "absence of explosion", rather than "eventually forgetting what the initial conditions were". The latter is better quantified using *incremental* settings.

Let us call a DT system $S \in \mathcal{S}^{m,k}_{DT}(X)$ *L2 incrementally stable* when

$$\sum_{t=0}^{\infty} |y_1(t) - y_2(t)|^2 < \infty \tag{2.10}$$

for every $w \in \ell^m$, $x_1, x_2 \in X$, $y_1 \in S(w, x_1)$, $y_2 \in S(w, x_2)$.

Similarly, a CT system $S \in \mathcal{S}_{CT}^{m,k}(X)$ is called *L2 incrementally stable* when

$$\int_0^\infty |y_1(t) - y_2(t)|^2 dt < \infty \tag{2.11}$$

for every $w \in \mathcal{L}^m$, $x_1, x_2 \in X$, $y_1 \in S(w, x_1)$, $y_2 \in S(w, x_2)$.

Incremental L2 stability guarantees that changing initial conditions produces only a vanishing perturbation in system response, where "vanishing" means "having finite energy". Note that the incremental stability, as defined here, is only concerned with relative difference between responses to different initial conditions, and hence does not imply finite L2 gain stability.

2.7 Interconnections

Interconnections represent a simple way of combining several systems (called *subsystems* in this case) to define a single one by declaring simple relations between inputs and outputs of subsystems.

2.7.1 Series Interconnection

For two DT systems $S_1 \in \mathcal{S}_{DT}^{m,q}(X_1)$ and $S_2 \in \mathcal{S}_{DT}^{q,k}(X_2)$, the *series interconnection* $S = S_2 \circ S_1$ is the system $S \in \mathcal{S}_{DT}^{m,k}(X_1 \times X_2)$ with boundary conditions set

$$X_1 \times X_2 \overset{\text{def}}{=} \{(x_1, x_2) : \quad x_1 \in X_1, \;\; x_2 \in X_2\},$$

for which input $w \in \ell^m$ and boundary condition $x = (x_1, x_2)$ produce responses which are responses of S_2 to signal $u \in \ell^q$ with boundary condition x_2, where u is a response of S_1 to input w and boundary condition x_1, i.e.

$$S(w, (x_1, x_2)) = \{y \in S_2(u, x_2) : u \in S_1(w, x_1)\}. \tag{2.12}$$

It is important to note that a series interconnection is always *well-posed*, i.e. guarantees

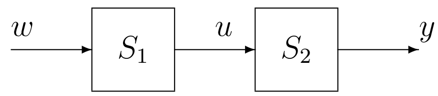


Figure 2.3: Series interconnection

to define a true system, S ,in the sense that the set $S(w,(x_1,x_2))$, as defined in (2.12), is not empty for every $w \in \ell^m$, $x_1 \in X_1$, $x_2 \in X_2$ (as long as S_1 and S_2 are valid systems, in the sense that the sets $S_1(w,x_1)$ and $S_2(u,x_2)$ are not empty for all $w \in \ell^m$, $u \in \ell^q$, $x_1 \in X_1$, $x_2 \in X_2$).

The definition in the CT case are similar, with ℓ-spaces replaced by $\mathcal{L}$-spaces.

2.7.2 Difference Interconnection

For two DT systems $S_1 \in \mathcal{S}_{DT}^{m,k}(X_1)$ and $S_2 \in \mathcal{S}_{DT}^{m,k}(X_2)$, the *series interconnection* $S = S_1 - S_2$ is the system $S \in \mathcal{S}_{DT}^{m,k}(X_1 \times X_2)$ defined by

$$S(w,(x_1,x_2)) = \{y_1 - y_2 : \quad y_1 \in S_1(w,x_1),\ y_2 \in S(w,x_2)\}. \tag{2.13}$$

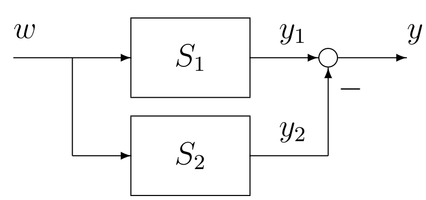


Figure 2.4: Difference interconnection

Just like any series interconnection, a difference interconnection is guaranteed to be well-posed. The CT definition is similar to the one given for the DT case.

2.7.3 Unity Feedback Interconnection

For a DT system $S \in \mathcal{S}_{DT}^{m,m}(X)$, its *unity feedback interconnection* is the system $S^\circ \in \mathcal{S}_{DT}^{m,m}(X)$ defined by

$$S^\circ(w,x_0) = \{y : \quad y \in S(w+y, x_0 b)\}. \tag{2.14}$$

(The CT case definition is similar.)

It is important to understand that, in general, well-posedness of a unity feedback interconnection is not assured, i.e. the resulting sets $S^\circ(w,x_0)$ could be empty.

Example 2.5. If $S \in \mathcal{S}_{DT}^{m,m}(\{0\})$ is the "multiplication by a" system

$$S(w,0) = \{y : y(t) = aw(t)\},$$

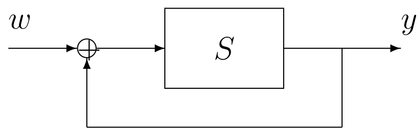


Figure 2.5: Unity feedback interconnection

where $a \in \mathbb{R}$ is a parameter, every $y \in S^\circ$ must satisfy $y = w + ay$. When $a \neq 1$, this defines a valid system

$$S^\circ(w, \{0\}) = \left\{ y : \quad y(t) = \frac{1}{1-a} w(t) \right\}.$$

However, when $a = 1$, the equations for S° require $w \equiv 0$, which means that (2.14) defines an empty set for all w except $w \equiv 0$. Hence, the unity feedback interconnection is not well-posed when $a = 1$.

Here is a slightly more interesting example.

Example 2.6. Consider the task of stabilizing the double integrator system

$$\ddot{y} = u,$$

where y represents the controlled output, and u is the control signal, restricted to taking values in the $\{-1, 0, 1\}$ set. It is tempting to consider feedback law like this:

$$u(t) = -\mathrm{sign}(y(t) + \dot{y}(t)), \quad \text{where} \quad \mathrm{sign}(r) = \begin{cases} 1, & r > 0, \\ 0, & r = 0, \\ -1, & r < 0. \end{cases}$$

On the surface, the resulting feedback system looks good: a simple differentiation shows that, subject to system equations, the function

$$V(y(t), \dot{y}(t)) = |y(t) + \dot{y}(t)| + 0.5\dot{y}(t)^2$$

is guaranteed to decrease at rate 1 for all $t \geq 0$ with $y(t) + \dot{y}(t) \neq 0$. Since the only solution of the system equations which has $y(t) + \dot{y}(t) = 0$ for a positive length of time is $y(t) \equiv 0$, $\dot{y}(t) \equiv 0$, this suggest a conclusion that the zero equilibrium is reached in finite time:

$$y(t) = 0 \quad \text{for all} \quad t > |y(0) + \dot{y}(0)| + 0.5|\dot{y}(0)|^2.$$

The conclusion is not quite correct, because, in fact, the set of system equations does not have a solution extending (in time) beyond a point where

$$0 < |y(t) + \dot{y}(t)| < 1.$$

In terms of the standard format of system modeling used in this class, the open loop system can be represented as the series interconnection of $S \in \mathcal{S}_{CT}^{1,1}(\mathbb{R}^2)$, defined by

$$S(w,[v_0;y_0]) = \left\{ y+v: \quad v(t) = v_0 + \int_0^t w(\tau)d\tau,\ y(t) = y_0 + \int_0^t v(\tau)d\tau \right\},$$

and $\Delta \in \mathcal{S}_{CT}^{1,1}(\{0\})$, defined by

$$\Delta(e,0) = \{u: \ u(t) = -\text{sign}(e(t))\}.$$

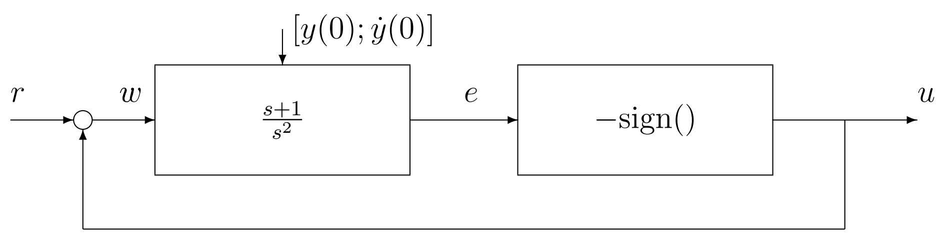


Figure 2.6: Switching feedback

This is supposed to result in a feedback system $F \in \mathcal{S}_{CT}^{1,1}(\mathbb{R}^2)$ with input r, boundary conditions set $\mathbb{R}^2$, and output u, but the interconnection is not well-posed, so no feedback system is actually defined.