Chapter 3

Small Gain Theorem and Integral Quadratic Constraints

This chapter presents a version of the "small gain theorem" for L2 gains, and then proceeds to introduce the general technique of working with quadratic "dissipation inequalities".

3.1 Small Gain Theorems for L2 Gains

There exist many versions of the "small gain theorem", distinguished by the type of "gain" and the type of interconnection they apply to. One of the simplest versions applies to the unit feedback interconnection on Figure 3.1, and claims that, assuming well-posedness, if system Δ has L2 gain $\gamma < 1$, then the unity feedback interconnection system Δ° (input w, output y) has L2 gain not larger than $\gamma/(1-\gamma)$.

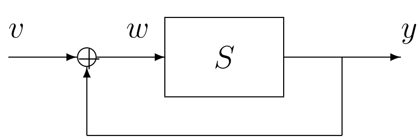

Figure 3.1: Unity feedback interconnection

The "small gain" statement applies to DT and CT systems, and to both "asymptotic" and "fixed state" versions of L2 gain. Of course, if the assumption about Δ is in terms of an "asymptotic" L2 gain then the conclusion about Δ° is also in terms of an "asymptotic" L2 gain.

3.1.1 The Fixed State Small L2 Gain Theorem

Here is a formal presentation of a one version of the small gain theorem (the rest are formulated and proven in a similar way).

Theorem 3.1. *If system $\Delta \in \mathcal{S}_{DT}^{m,m}(X)$ has fixed state L2 gain $\gamma < 1$, and the unity feedback system $\Delta^\circ \in \mathcal{S}_{DT}^{m,m}(X)$ is well defined then the fixed state L2 gain of Δ° is not larger than $\gamma/(1-\gamma)$.*

Example 3.1. When $\Delta \in \mathcal{S}_{DT}^{1,1}(\mathbb{R})$ is the "one step delay times a" system defined by

$$\Delta(w, x_0) = \{y : \quad y(0) = x_0,\ y(t+1) = aw(t)\ \forall\ t \in \mathbb{Z}_+\},$$

the unity feedback system Δ° is well defined for all values of parameter $a \in \mathbb{R}$, and represents the difference equation

$$y(t+1) = ay(t) + v(t), \quad y(0) = x_0.$$

The fixed state L2 gain of Δ is $\gamma = |a|$. It can be shown (using the results on computing L2 gains of LTI models, to be discussed in the next chapter) that the fixed state L2 gain of Δ° is $\gamma^\circ = 1/(1-|a|)$ for $|a| < 1$, and $\gamma^\circ = \infty$ for $|a| \geq 1$, which shows that the closed loop L2 gain bound from Theorem 3.1 is exact in this case.

There are several ways to approach the proof of Theorem 3.1. To stress the generalizations to be discussed later in the chapter, we start with the following statement about specific quadratic forms.

Lemma 3.1. *For every $h \in [0, 1)$ there exists $d \geq 0$ such that*

$$\frac{h^2}{(1-h)^2}|w-y|^2 - |y|^2 - d(h^2|w|^2 - |y|^2) \geq 0 \quad \forall\ w, y \in \mathbb{R}^m. \tag{3.1}$$

Proof of Lemma 3.1

The left hand side of the inequality in (3.1) is a quadratic form $\sigma = \sigma(x)$ with respect to the vector $x = [w; y] \in \mathbb{R}^{2m}$:

$$\sigma(x) = x'Qx \quad \text{where} \quad Q = \begin{bmatrix} \left(\frac{h^2}{(1-h)^2} - dh^2\right) I_m & -\frac{h^2}{(1-h)^2} I_m \\ -\frac{h^2}{(1-h)^2} I_m & \left(d - 1 + \frac{h^2}{(1-h)^2}\right) I_m \end{bmatrix}$$

(I_m denotes the m-by-m identity matrix, and the prime ' means "Hermitean conjugation", which is the same as transposition for real vectors or matrices, and the elements of $\mathbb{R}^n$ are interpreted as

n-by-1 real matrices). By construction, (3.1) is equivalent to $Q \geq 0$ being a positive semidefinite matrix. Using the fact that 2-by-2 block matrix

$$Q = \begin{bmatrix} a & b \\ b' & c \end{bmatrix}$$

is positive semidefinite if

$$c = c' > 0, \quad a = a' \geq bc^{-1}b',$$

the statement follows by using $d = 1/(1-h)$. ■

Proof of Theorem 3.1

Consider arbitrary $h \in (\gamma, 1)$, $T \in \mathbb{Z}_+$, and $y \in \Delta^\circ(w, x_0)$ (which also means $y \in \Delta(w+y, x_0)$), substitute $y = y(t)$, $w = w(t)$ into the inequality from (3.1), and then take a sum from $t = 0$ to $t = T$, to get

$$\sum_{t=0}^{T} \left\{ \bar{h}^2 |w(t) - y(t)|^2 - |y(t)|^2 \right\} \geq d \sum_{t=0}^{T} \left\{ h^2 |w(t)|^2 - |y(t)|^2 \right\}, \tag{3.2}$$

where $\bar{h} = h/(1-h)$. Since the fixed state L2 gain of Δ is less than h, the sum on the right side of (3.2) is bounded from below by a constant $c = c(x_0)$ depending on the boundary condition x_0 only. Since $d \geq 0$, this means that the sum on the left side of (3.2) is bounded from below by the constant $c^\circ = dc(x_0)$ which depends on the boundary condition x_0 only. Since $w - y = v$, this means that the fixed state L2 gain of Δ° is not larger than $\bar{h} = h/(1-h)$ for every $h \in (\gamma, 1)$. As $h \to \gamma$, we conclude that the fixed state L2 gain of Δ° is not larger than $\gamma/(1-\gamma)$. ■

3.2 Integral Quadratic Constraints

Given a system $S \in \mathcal{S}^{m,k}(X)$ and a quadratic form $\sigma : \mathbb{R}^{m+k} \to \mathbb{R}$, let us say that *$S$ satisfies the fixed state integral quadratic constraint (IQC) defined by σ* if for every $x_0 \in X$ there exists a constant $c = c(x_0)$ such that the inequality

$$\sum_{t=0}^{T} \sigma([w(t); y(t)])dt \geq c \tag{3.3}$$

is satisfied for all $w \in \ell^m$ and $y \in S(w, x_0)$.

The proof of Theorem 3.1 follows a certain path. First, the assumptions and the conclusion to be proven are viewed as IQCs: the original L2 gain bound means that Δ satisfies the IQC defined by

$$\sigma_1([w; y]) = h^2|w|^2 - |y|^2$$

while the objective is to prove that Δ satisfies the IQC defined by

$$\sigma_0([w;y]) = r|w-y|^2 - |y|^2$$

for r as small as $h/(1-h)$.

The second step is pure manipulation with parameter-dependent quadratic forms: we consider the function $\sigma: \mathbb{R}^{2m} \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\sigma([w;y],[r;d]) = \sigma_0([w;y]) - d\sigma_1([w;y]) = r|w-y|^2 - |y|^2 - d(h^2|w|^2 - |y|^2) \qquad (3.4)$$

(this $\sigma = \sigma(v,x)$ is a quadratic form with respect to its first argument $v = [w;y] \in \mathbb{R}^{2m}$, and an affine function of its second argument $x = [r;d] \in \mathbb{R}^2$), and try to find the value of the second argument $x = [r;d]$ such that $d \geq 0$, and the resulting quadratic form of the first argument $v \in \mathbb{R}^{2m}$ is positive semidefinite. For every r, d satisfying these conditions, $\sqrt{r}$ is an upper bound for the L2 gain of Δ°.

Since we want the smallest possible upper bound for the closed loop L2 gain, the objective is to minimize r subject to the constraints. In the proof of Theorem 3.1 , this is done "analytically" (in fact, the algebra behind the optimization is hidden, and the explicit optimizer $r = h/(1-h)$, $d = 1/(1-h)$ is given). Similar "analytical" proofs can be given to many similar statements concerning stability of systems described in terms of IQC. An alternative, once a specific value of h is given, would be to solve the optimization task

$$r \to \min \quad \text{subject to} \quad d \geq 0, \quad \sigma(v,[r,d]) \geq 0 \ \forall\ v$$

numerically, using appropriate optimization software, to be discussed later in this chapter.

Of course, this approach is not limited to the fixed state IQC in discrete time. Similar definitions are available in the "asymptotic" format (where the constant c in (3.3) is allowed to depend on w and y, but *not* on T, i.e. $c = c(x_0, w, y)$), and for CT models (where the sums from $t = 0$ to $t = T$ are replaced by integrals from $t = 0$ to $t = T$).

3.2.1 Example: Passive Feedback

The following classical statement on feedback system stability is another example of using IQCs. For variety, it will be presented in an "asymptotic" continuous time setting.

Let us call system $\Delta \in \mathcal{S}_{CT}^{m,m}(X)$ *passive* if

$$\inf_{T \geq 0} \int_0^T y(t)'w(t)dt > -\infty \qquad (3.5)$$

for every $w \in \mathcal{L}^m$, $x_0 \in X$, and $y \in \Delta(w, x_0)$. The terminology is motivated by the situation when the components of y are currents, and the corresponding components of

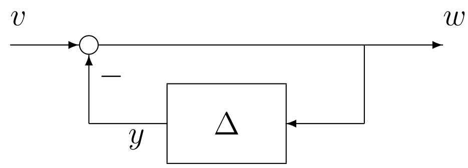

Figure 3.2: Interconnection for Theorem 3.2

w are voltages, at all ports of a passive circuit. Then the integral in (3.5) represents the energy supplied into the circuit on the time interval $[0, T]$, which natuarally is bounded from below by minus the initial value of energy stored within the circuit.

It turns out that, when well-posed, a feedback interconnection with a passive circuit subsystem Δ, as shown on Figure 3.2, has L2 gain not larger than 1.

Theorem 3.2. *If system $\Delta \in \mathcal{S}_{CT}^{m,m}(X)$ is passive, and the system*

$$\Delta^+(v, x_0) = \{w : \quad v - w \in \Delta(w, x_0)\}$$

is well defined, then the L2 gain of Δ^+ isnot larger than 1

Proof of Theorem 3.2

By assumption, the asymptotic IQC defined by

$$\sigma_1([w; y]) = w'y$$

is satisfied for Δ. Since $v = w + y$, the objective is to prove that Δ satisfies the IQC defined by

$$\sigma_0([w; y]) = |w + y|^2 - |w|^2.$$

Indeed, the latter IQC is certified by the fact that the quadratic form $\sigma = \sigma_0 - 2\sigma_1$, i.e.

$$\sigma([w; y]) = |w + y|^2 - |w|^2 - 2w'y = |y|^2$$

is positive semidefinite. ■

3.3 IQCs and Semidefinite Programming

One convenient way of setting up an *optimization problem* (or "program") is by using the format

$$Lx \to \min_x \quad \text{subject to} \quad x \in Q, \quad Ax = B, \tag{3.6}$$

where Q is typically a relatively simple subset Q of $\mathbb{R}^m$, A is a real m-by-n matrix, L is a 1-by-m row vector, and $B \in \mathbb{R}^n$. The task is usually to figure out whether the program is *feasible*, i.e. if an element $x \in Q$ such that $Ax = B$ does exist, and, if feasible, finding an element $x \in Q$ such that $Ax = B$ and Lx is minimal (or at least close to minimal in some sense).

For example, a linear program can be defined in the form (3.6) with Q being the "positive quadrant" $Q = \mathbb{R}^m_+$, i.e. the subset of vectors $x \in \mathbb{R}^m$ with non-negative coefficients.

Let us fix a particular way ϕ of re-arranging the entries of m-by-1 column vectors, where $m = k(k+1)/2$, to form real symmetric k-by-k matrices. For example, we can have

$$\phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}.$$

Let $\mathbb{S}^k_+$ be the set of all vectors $x \in \mathbb{R}^m$ for which $\phi(x)$ is a positive semidefinite matrix. When $Q = \mathbb{S}^k_+$ for some k, the task (3.6) is called a *semidefinite program*.

This definition is general enough, but it turns out to be both convenient and more efficient to view Q as the set of vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix}, \quad x_i \in Q_i,$$

where each of the Q_i is, for some $d = d_i$, one of the following:

(a) $\mathbb{R}^d$;

(b) $\mathbb{R}^d_+$;

(c) $\mathbb{S}^d_+$;

(d) the set

$$\mathbb{L}^k = \left\{\begin{bmatrix} x_0 \\ \dots \\ x_k \end{bmatrix} \in \mathbb{R}^{k+1} : \quad x_0 \geq 0,\ x_0^2 \geq x_1^2 + \dots + x_k^2\right\};$$

(e) the set

$$\mathbb{L}^k = \left\{\begin{bmatrix} x_0 \\ \dots \\ x_k \end{bmatrix} \in \mathbb{R}^{k+1} : \quad x_0 \geq 0,\ x_1 \geq 0,\ x_0 x_1 \geq x_2^2 + \dots + x_k^2\right\}.$$

For example, the task of finding real numbers r, d such that $d \geq 0$, the quadratic form σ in (3.4) is positive definitene, and r is as small as possible (something motivated by the small gain theorem), can be viewed as a semidefinite program with

$$Q = \left\{ \begin{bmatrix} r \\ d \\ s \end{bmatrix} : \quad r \in \mathbb{R},\ d \in \mathbb{R}_+,\ s \in \mathbb{S}^2 \right\}.$$

In the resulting task (3.6) we have $n = d(d+1)/2$ with $d = 2$,

$$B = 0, \quad A \begin{bmatrix} r \\ d \\ s \end{bmatrix} = \phi^{-1}(\sigma - \phi(s)) = \begin{bmatrix} r - dh^2 & -r \\ -r & r - 1 + d \end{bmatrix} - \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$$

(so that $Ax = B$ simply states that σ is positive semidefinite), and

$$L \begin{bmatrix} r \\ d \\ s \end{bmatrix} = r.$$

3.3.1 The `spot` Toolbox

There are many tools for solving semidefinite programs numerically. Currently, withing the MATLAB environment, one of the most commonly used free semidefinite program solvers is `SeDuMi`. Since representing semidefinite programs in the `SeDuMi`-prescribed format can be time-consuming and error-prone, additional tools were developed to help with problem description (in particular, the free YALMIP toolbox appears to be popular).

This section describes the use of an alternative MATLAB toolbox, `spot`, which will be made available from the 6.245 class locker. Here is the code for solving the semidefinite program associated with the small gain theorem. It returns the best upper bound for the closed loop L2 gain, when stability can be establishes (subject to the well-posedness assumption), or `[]`.

```
function g=spotsample0(h)
r=msspoly('r');            % symbolic [r]
d=msspoly('d');            % symbolic [d]
s=msspoly('s',[3,1]);      % symbolic [s1;s2;s3]
pr = mssprog;              % initialize program
pr.free = r;               % register r as free
```

```
pr.pos = d;              % register d as positive
pr.psd = s;              % register s as semidefinite
S = mss_v2s(s);          % S=[s1 s2;s2 s3]
pr.eq = [r-d*(h^2) -r; -r r-1+d]-S;   % this is 0
pr.sedumi = r;           % minimize r
g=sqrt(pr({r}));         % get optimum
```

Here is a different code, which is mostly indented to save programmer's brains by skipping some algebra done by hand (it refers directly to quadratic form (3.4) instead of its matrix). In this case, it turns out that it also generates a smaller size `SeDuMi` program.

```
function g=spotsample1(h)
r=msspoly('r');          % symbolic [r]
d=msspoly('d');          % symbolic [d]
s=msspoly('s',[3,1]);    % symbolic [s1;s2;s3]
w=msspoly('w');          % symbolic [r]
y=msspoly('y');          % symbolic [y]
z=[w;y];
pr = mssprog;            % initialize program
pr.free = r;             % register r as free
pr.pos = d;              % register d as positive
pr.psd = s;              % register s as semidefinite
S = mss_v2s(s);          % S=[s1 s2;s2 s3]
% equality constraint
pr.eq = r*(w-y)^2-y^2-d*(h^2*w^2-y^2)-z'*S*z;
pr.sedumi = r;           % minimize r
g=sqrt(pr({r}));         % get optimum
```

Note the difference between *registered* variables `r,d,s` and *unregistered* variables `w,y`. When handling equality constraints which involve unregistered variables, `spot` generates equality constraints which are equivalent to the expression being zero for *all* possible values of unregistered variables.

Example: Robustness to Structured Uncertainty

Consider the feedback interconnection from Figure 3.3, where v, w, q, y are signals from $\mathcal{L}^m$, $M\nabla\mathcal{S}_{CT}^{m,m}(\{0\})$ is the linear system defined by a real given m-by-m real matrix M_0:

$$M(w,0) = \{M_0 w\},$$

and $\Delta \in \mathcal{S}_{CT}^{m,m}(X)$ is the *parallel connection* of m systems $\Delta_i \in \mathcal{S}_{CT}^{1,1}(X_i)$ $(i = 1, \dots, m)$, in the sense that $X = X_1 \times \cdots \times X_m$, and

$$\Delta\left(\begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix}, (x_1, \dots, x_m)\right) = \left\{\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} : \quad y_i \in \Delta_i(q_i, x_i)\right\}.$$

Assume furthermore that L2 gain of each of the subsystems Δ_i is known to be smaller than 1. The objective is to establish L2 gain stability of the closed loop (as usually, subject to the well-posedness assumption).

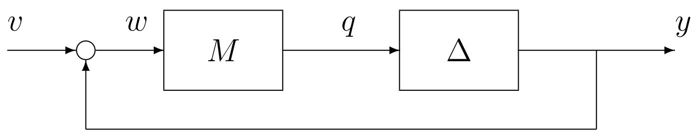

Figure 3.3: M/Δ interconnection

Since the L2 gain bound for each Δ_i establishes that $\Delta \circ M$ satisfies the IQC defined by

$$\sigma_i([w; y]) = |e_i M w|^2 - |e_i y|^2 \quad (i = 1, 2, \dots, m),$$

where e_i is the 1-by-m row vector with all "0" elements except "1" at the i-th position. On the other hand, the closed loop L2 gain is not larger than $\gamma \geq 0$ if $\Delta \circ M$ satisfyies the IQC defined by

$$\sigma_0([w; y]) = r|w - y|^2 - |y|^2$$

with $r = \gamma^2$.

The standrad approach to IQC analysis suggests searching for non-negative constants $d_1, \dots, d_m$ and $r \in \mathbb{R}$ such that the quadratic form

$$\sigma = \sigma_0 - \sum_{i=1}^{m} d_i \sigma_i$$

is positive semidefinite. To simplify notation, it is useful to note that

$$\sum_{i=1}^{m} d_i \sigma_i([w; y]) = (Mw)' D (Mw) - y' D y,$$

where

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{bmatrix}$$

is the diagonal matrix with entries d_i on the diagonal.

The following `spot` code implements minimization of r subject to the constraints.

```
function r=spotsample(M)
% function r=spotsample(M)
% M must be a real m-by-matrix
m=size(M,1);                % problem dimension
r=msspoly('r');             % [r]
w=msspoly('w',[m,1]);       % [w1;w2;...;wm]
y=msspoly('y',[m,1]);       % [y1;y2;...;ym]
d=msspoly('d',[m,1]);       % [d1;d2;...;dm]
q=msspoly('Q',nchoosek(2*m+1,2));  %[q1;...]
pr = mssprog;               % initialize program
pr.free = r;                % register r
pr.pos = d;                 % register d
pr.psd = q;                 % register q
D = diag(d);                % diagonal D
Q = mss_v2s(q);             % symmetric Q
z = [w;y];
v = w-y;
Mw = M*w;
pr.eq = r*v'*v-y'*y-Mw'*D*Mw+y'*D*y-z'*Q*z;
pr.sedumi = r;              % minimize r
r=pr({r});                  % L2 gain bound
```