Chapter 4

Transfer Function Models

This chapter introduces models of linear time invariant (LTI) systems defined by their *transfer functions* (or, in general, *transfer matrices*). The subject is expected to be familiar to the reader, especially in the case of *rational* transfer functions, which are suitable for describing systems defined in terms of ordinary differential equations and difference equations. The main objective of the chapter is to build a mathematical framework suitable for handling the non-rational transfer functions resulting from partial differential equation models which are stabilizable by finite order LTI controllers.

4.1 Fourier Transforms and the Parseval Identity

Fourier transforms play a major role in defining and analyzing systems in terms of non-rational transfer functions. This section requires some background in the theory of integration of functions of a real argument (measureability, Lebesque integrabilty, completeness of L_2 spaces, etc.), and presents some minimal technical information about Fourier transforms for "finite energy" functions on $\mathbb{Z}$ and $\mathbb{R}$.

4.1.1 Background: Spaces of Square Integrable Functions

This chapter deals extensively with complex vector-valued square integrable functions defined on the sets $\mathbb{R}$, $j\mathbb{R}$ (the "imaginary axis" $\{s \in \mathbb{C} : \ \mathrm{Re}\{s\} = 0)$, $\mathbb{T}$ (the unit circle $\{z \in \mathbb{C} : \ |z| = 1\}$), and $\mathbb{Z}$ (all integers). The sets of all such functions (traditionally called "spaces") will be denoted by $L_2^m(X)$, where $X \in \{\mathbb{R}, j\mathbb{R}, \mathbb{T}, \mathbb{Z}\}$.

While there are some technical distinctions caused by the differences between the four domains X of interest ($\mathbb{R}$, $j\mathbb{R}$, $\mathbb{T}$, and $\mathbb{Z}$), the sets $L_2^m(X)$ share important abstract

features, which allow one to treat them in a way similar to how ordinary vector spaces $\mathbb{C}^n$ are treated. Specifically, all these sets are *complex vector spaces*, i.e. operations of addition and scaling by a complex scalar are defined on $L_2^m(X)$, and satisfy the usual commutative and distributive laws. Moreover the spaces have an *inner product* operation, mapping every ordered pair (f, g) of functions $f, g \in L_2^m(X)$ to a complex number c denoted by $f'g$. The inner prodiuct satisfies the conditions of *Hermitean symmetry* ($g'f$ is the complex conjugate of $f'g$), linearity with respect to the second argument ($f'(cg) = c(f'g)$ for $c \in \mathbb{C}$, and $f'(g_1+g_2) = f'g_1+f'g_2$), and positive semidefiniteness ($f'f \geq 0$). The function $|f| = \sqrt{f'f}$ is used as a measure of *length* of a function, and satisfies the *triangle inequality* $|f+g| \leq |f|+|g|$ (or, equivalently, the Cauchy inequality $|f'g| \leq |f|\cdot|g|$), so that one can talk about *convergence* in $L_2^m(X)$: a sequence $\{f_k\}_{k=0}^{\infty}$ of functions $f_k \in L_2^m(X)$ is said to converge to $f \in L_2^m(X)$ when $f_k - f| \to 0$ as $k \to \infty$. With respect to this definition of convergence, the sets $L_2^m(X)$ have the *completeness* property: for every sequence $\{f_k\}_{k=0}^{\infty}$ of functions $f_k \in L_2^m(X)$ satisfying the *Cauchy* property ($|f_k - f_i| \to 0$ as $k, i \to \infty$) there exists a function $f \in L_2^m(X)$ such that $|f_k - f| \to 0$ as $k \to \infty$.

For example, $L_2^m(\mathbb{Z})$ will denote the set of all functions $u : \mathbb{Z} \to \mathbb{C}^m$ (essentially, double-sided sequences of vectors from $\mathbb{C}^m$) such that

$$\sum_{t\in\mathbb{Z}} |u(t)|^2 < \infty,$$

with the inner product

$$u'v \stackrel{\text{def}}{=} \sum_{t\in\mathbb{Z}} u(t)'v(t).$$

Similarly, $L_2^m(\mathbb{R})$ will be the set of all *measureable* functions $u : \mathbb{R} \to \mathbb{C}^m$ such that

$$\int_{-\infty}^{\infty} |u(t)|^2 dt < \infty,$$

with the inner product

$$u'v \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} u(t)'v(t)dt.$$

Note that a function $f \in L_2^m(\mathbb{R})$ such that $|f| = 0$ can take non-zero values, but in this case the *Lebesque measure* of the set $\{t : f(t) = 0\}$ is zero.

A function $F : \mathbb{T} \to \mathbb{C}^k$ is called *integrable* when the function $\tilde{F} : (-\pi, \pi) \to \mathbb{C}^m$ defined by $\tilde{F}(\Omega) = F(\exp(j\Omega))$ is Lebesque integrable, in which case we use the shortcut notation

$$\int_{\mathbb{T}} F(z)dm(z) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\exp(j\Omega))d\Omega.$$

The set $L_2^m(\mathbb{T})$ contains integrable functions $U:\ \mathbb{T} \to \mathbb{C}^m$ such that

$$\int_{\mathbb{T}} |U(z)|^2 dm(z) < \infty.$$

The inner product on $L_2^m(\mathbb{T})$ is defined by

$$U'V = \int_{\mathbb{T}} U(z)'V(z)dm(z).$$

Finally, $L_2^m(j\mathbb{R})$ contains functions $U:\ j\mathbb{R} \to \mathbb{C}^m$ such that the function $g:\ \mathbb{R} \to \mathbb{C}^m$ defined by $g(\omega) = U(j\omega)$ is square integrable. The inner product on $L_2^m(j\mathbb{R})$ is defined by

$$U'V = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(j\omega)'V(j\omega)d\omega.$$

4.1.2 Fourier Transform on $\mathbb{Z}$

For a function (sequence) $u:\ \mathbb{Z} \to \mathbb{C}^m$, one expects the Fourier transform to be a function $U:\ \mathbb{T} \to \mathbb{C}^m$ given by

$$U(z) = \sum_{t\in\mathbb{Z}} u(t)z^{-t}. \tag{4.1}$$

This definition is straightforward for *absolutely summable sequences*, i.e. those satisfying the condition

$$\sum_{t\in\mathbb{Z}} |u(t)| < \infty,$$

which guarantees that the sum in (4.1) converges uniformly over $z \in \mathbb{T}$. Moreover, the simple observation that

$$\int_{\mathbb{Z}} \bar{z}^i z^k dm(z) = \delta_{ik} \stackrel{\text{def}}{=} \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases} \tag{4.2}$$

for all $i, k \in \mathbb{Z}$ yields the ever helpful *Parseval identity*

$$\sum_{t\in\mathbb{Z}} |u(t)|^2 = \int_{\mathbb{T}} |U(z)|^2 dm(z). \tag{4.3}$$

The problem is, the class of summable seqences u is too narrow to be useful in many applications. The objective of this subsection is to introduce Fourier transform for *square summable* sequences, i.e. elements of $L_2^m(\mathbb{Z})$.

According to a "proper" definition, the *Fourier transform* U of $u \in L_2^m(\mathbb{Z})$ is any $U \in L_2^m(\mathbb{T})$ such that $|U_T - U| \to 0$ as $T \to \infty$, where

$$U_T(z) = \sum_{t=-T}^{T} u(t) z^{-t}. \tag{4.4}$$

According to this definition, a given sequence $u \in L_2^m(\mathbb{Z})$ can have multiple Fourier transforms, but this non-uniqueness is not really essential, as any two Fourier transforms U, V of the same square summable sequence u must satisfy

$$|U - V| = |(U - U_T) - (V - U_T)| \le |U - U_T| + |V - U_T| \to 0 \ \text{ as } \ T \to \infty,$$

which means that $|U - V| = 0$, i.e. $U(z) - V(z) = 0$ for almost all z, so that, for all practical purposes, U and V can be considered equal.

The main statement about Fourier transforms of square summable sequences is their existence, coupled with the Parseval identity.

Theorem 4.1. *Every $u \in L_2^m(\mathbb{Z})$ has a Fourier transform $U \in L_2^m(\mathbb{T})$. Moreover, every Fourier transform $U \in L_2^m(\mathbb{T})$ of $u \in L_2^m(\mathbb{Z})$ satisfies (4.3).*

Proof. For arbitrary positive integers $\tau < T$ the Parseval identity yields

$$\|U_T - U_\tau\|_2^2 = \sum_{t=\tau}^{T-1} |u(t)|^2 \le \sum_{t=\tau}^{\infty} |u(t)|^2 \to 0 \ \text{ as } \ \tau \to \infty,$$

where U_T are defined in (4.4). Hence $\{U_T\}$ is a Cauchy sequence, and, by *completeness* of $L_2^m(\mathbb{T})$, has a limit U. By definition, U is a Fourier transform of u.

To prove the Parseval identity, note that

$$|U| - |U_T| = |(U - U_T) + U_T| - |U_T| \le |U - U_T| + |U_T| - |U_T| \to 0 \ \text{ as } \ T \to \infty,$$

$$|U_T| - |U| = |U_T - U + U| - |U| \le |U_T - U| + |U| - |U| \to 0 \ \text{ as } \ T \to \infty.$$

Hence

$$|U|^2 = \lim_{T\to\infty} |U_T|^2 = \lim_{T\to\infty} \sum_{t=-T}^{T} |u(t)|^2 = |u|^2.$$

■

4.1.3 Approximations by Trigonometric Polynomials in $L_2^m(\mathbb{T})$

A *trigonometric polynomial* (of a single argument) is a continuous function $P: \mathbb{T} \to \mathbb{C}^m$ which can be expressed in the form

$$P(z) = \sum_{t=-T}^{T} p(t) z^{-t},$$

where $p(t) \in \mathbb{C}^m$ are some vectors of coefficients. (The name is likely due to the fact that z^n is a polynomial with respect to $cos(\Omega)$ and $\sin(\Omega)$ when $z = \exp(j\Omega)$.) For the development of Fourier transforms, we need the classical theorem establishing that trigonometric polynomials can approximate continuous functions $V: \mathbb{T} \to \mathbb{C}^m$ arbitrarily well.

Theorem 4.2. *For every continuous function $V: \mathbb{T} \to \mathbb{C}^m$ and every $\epsilon > 0$ there exists a trigonometric polynomial $P: \mathbb{T} \to \mathbb{C}^m$ such that*

$$\max_{z \in \mathbb{T}} |P(z) - V(z)| < \epsilon.$$

Proof. For every positive integer n define function $P_n: \mathbb{T} \to \mathbb{C}^m$ by

$$P_n(z) = \int_{\mathbb{T}} h_n(z, w) V(w) dm(w), \tag{4.5}$$

where

$$h_n(z, w) = \frac{1}{n} \left| 1 + \frac{w}{z} + \frac{w^2}{z^2} + \cdots + \frac{w^{n-1}}{z^{n-1}} \right|^2.$$

Since for every fixed n and w the function $z \to h_n(z, w)$ is a trigonometric polynomial, the result $P_n(z)$ of integration in (4.5) is a trigonometric polynomial, too.

The "kernel" function $h_n(z, w)$ has several useful properties:

(a) $h_n(z, w) \geq 0$;

(b) $\lim_{n\to\infty} \sup_{z:\ |z-w|>\delta} h_n(z, w) = 0$ for all $\delta \in (0, \pi)$;

(c) $\int_{\mathbb{T}} h_n(z, w) dm(w) = 1$ for all n.

These properties guarantee that

$$\lim_{n\to\infty} \max_{z \in \mathbb{T}} |P_n(z) - V(z)| = 0. \tag{4.6}$$

Indeed, since V is a continuous function, for every $\epsilon > 0$ there exists $\delta \in (0, \pi)$ such that

$$|V(w) - V(z)| < 0.5\epsilon \quad \text{whenever} \quad |w - z| < \delta.$$

For this $\delta > 0$, property (b) of the kernel guarantees existence of n such that

$$h_n(z,w) < \frac{0.5\epsilon}{1+2\gamma} \quad \text{for} \quad |z-w| > \delta, \quad \text{where} \quad \gamma = \max_{z\in\mathbb{T}} |V(z)|.$$

Hence for every $z \in \mathbb{T}$

$$\begin{aligned}
|P_n(z) - V(z)| &= \left| \int_{\mathbb{T}} h_n(z,w)(V(z)-V(w))dm(w) \right| \\
&\leq \int_{|z-w|<\delta} h_n(z,w)|V(z)-V(w)|dm(w) + \int_{|z-w|>\delta} h_n(z,w)|V(z)-V(w)|dm(w) \\
&\leq \int_{\mathbb{T}} 0.5\epsilon dm(w) + \int_{\mathbb{T}} \frac{0.5\epsilon}{1+2\gamma} 2\gamma dm(w) \\
&= \epsilon.
\end{aligned}$$

■

4.1.4 Fourier Coefficients

For every $U \in L_2^m(\mathbb{T})$ and $t \in \mathbb{Z}$ the "Fourier coefficients" integral

$$u(t) = \int_{\mathbb{T}} U(z) z^t dm(z) \tag{4.7}$$

defines a function (sequence) $u: \mathbb{Z} \to \mathbb{C}^m$.

The following statement, which relies on the result of the previous subsection, shows that Fourier coefficients deefine a function $U \in L_2^m(\mathbb{T})$ "almost completely".

Theorem 4.3. *If all Fourier coefficients (4.7) of a function $U \in L_2^m(\mathbb{T})$ are zero, then $|U| = 0$ (i.e. $U(z) = 0$ for almost all $z \in \mathbb{T}$).*

Proof. The theorem can be proven "component-wise", i.e. it is sufficient to consider the case $m = 1$. For $0 < \theta < \nu < 2\pi$ consider the function $F_{\theta,\nu}: \mathbb{T} \to \mathbb{C}$ defined by

$$F_{\theta,\nu}(\exp(j\Omega)) = \begin{cases} 1, & 0 \leq \Omega < \theta, \\ 1 - \frac{\Omega-\theta}{\nu-\theta}, & \theta \leq \Omega < \nu, \\ 0, & \nu \leq \Omega < 2\pi. \end{cases}$$

Since $F_{\theta,\nu}$ is continuous, there exists a sequence of trigonometric polynomials P_n such that

$$\lim_{n\to\infty} \max_{z\in\mathbb{T}} |F_{\theta,\nu}(z) - P_n(z)| = 0.$$

Hence

$$\lim_{n\to\infty}\int_{\mathbb{T}}(F_{\theta,\nu}(z)-P_n(z))V(z)dm(z)=0.$$

Since by assumption

$$\int_{\mathbb{T}}P(z)V(z)dm(z)=0$$

for every trigonometric polynomial P, we conclude that

$$\int_{\mathbb{T}}F_{\theta,\nu}V(z)dm(z)=\lim_{n\to\infty}\int_{\mathbb{T}}P_n(z)V(z)dm(z)=0.$$

For $0<\theta<2\pi$ let

$$F_\theta(\exp(j\Omega))=\begin{cases}1, & 0\le\Omega<\theta,\\ 0, & \theta\le\Omega<2\pi.\end{cases}.$$

Since

$$|F_\theta(z)-F_{\theta,\nu}(z)|\le 1 \quad\text{and}\quad \lim_{\nu\to\theta}F_{\theta,\nu}(z)=F_\theta(z),$$

we conclude that

$$\int_{\mathbb{T}}F_\theta(z)V(z)dm(z)=\lim_{\nu\to\theta}\int_{\mathbb{T}}F_\theta(z)V(z)dm(z)=0,$$

i.e.

$$\int_0^\theta V(\exp(j\Omega))d\Omega=0 \quad\text{for all}\quad \theta\in(0,2\pi).$$

According to the basic properties of Lebesque integration, this means that $V(\exp(j\Omega))=0$ or almost all $\Omega\in(0,2\pi)$. ■

4.1.5 Inverse Fourier Transform on $\mathbb{T}$

As stated by the following theorem, mapping (4.7) of a function $U\in L_2^m(\mathbb{T})$ to the sequence of its Fourier coefficients serves as the inverse of the Fourier transform (4.1).

Theorem 4.4. *For every $U\in L_2^m(\mathbb{T})$ the sequence $u:\ \mathbb{Z}\to\mathbb{C}^m$ defined by (4.7) has finite energy (i.e. $u\in\ell_2^m$). Moreover, U is a Fourier transform of u.*

Proof. First prove that u is square summable. For positive integers $T>0$ define functions $U_T:\ \mathbb{T}\to\mathbb{C}^m$ as in (4.4). Multiplying equality (4.7) by $u(t)'$ on the left and summing from $t=-T$ to $t=T$ yields

$$\sum_{t=-T}^{T}|u(t)|^2=\int_{\mathbb{T}}U_T(z)'U(z)dm(z).$$

Applying the Cauchy inequality $|U_T'U| \le |U_T| \cdot |U|$ to the inner product on the left side yields $|U_T|^2 \le |U| \cdot |U_T|$, hence $|U_T|^2 \le |U|^2$ for all $T > 0$, and therefore $u \in L_2^m(\mathbb{Z})$.

Let V denote the Fourier transform of u. Since $|U_T - V| \to 0$ as $T \to \infty$, and

$$\int_{\mathbb{Z}} z^t U_T(z) dm(z) = u(t) \quad \text{for} \quad |t| \le T,$$

U and V have same Fourier coefficients. Hence $|U - V| = 0$ (Theorem 4.3), which means that U is a Fourier transform of u as well. ■

4.1.6 Fourier Transform on $\mathbb{R}$

The derivation of Fourier transform on $L_2^m(\mathbb{R})$ follows a path similar to that of $L_2^m(\mathbb{Z})$. A key result is the following limited version of the Parseval identity.

Theorem 4.5. *The identity*

$$\int_{-T}^{T} |u(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |U_T(j\omega)|^2 d\omega, \tag{4.8}$$

where

$$U_T(j\omega) = \int_{-T}^{T} e^{-j\omega t} u(t) dt,$$

holds for every $T > 0$ and every function $u \in L_2^m(\mathbb{R})$.

Proof (a sketch). It is actually easier to prove the slightly more general identity

$$\int_{-T}^{T} u(t)' v(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} U_T(j\omega)' V_T(j\omega) d\omega, \tag{4.9}$$

where

$$U_T(j\omega) = \int_{-T}^{T} e^{-j\omega t} u(t) dt, \quad V_T(j\omega) = \int_{-T}^{T} e^{-j\omega t} v(t) dt,$$

and $u, v \in L_2^m(\mathbb{R})$ (proving the identity involves showing that $U_T, V_T \in L_2^m(j\mathbb{R})$). A good starting point is to consider functions u, v of the form

$$u(t) = \exp(j\pi a/T), \quad v(t) = \exp(j\pi b/T), \tag{4.10}$$

where a, b are arbitrary integer numbers. After some manipulations with integrals, this boils down to computing the integrals

$$\int_{-\infty}^{\infty} \frac{\sin^2(\omega)}{\omega(\omega + d\pi)} d\omega = \begin{cases} \pi, & d = 0, \\ 0, & d \in \mathbb{Z},\ d \ne 0. \end{cases}$$

(Would be easy to do this using the Parseval identity, but this would make for a "circular" argument: Parseval identity is exactly what is being proven right here!)

Once (4.9) is established for individual exponents (4.10), it extends easily to finite linear combinations of such exponents. According to the results on inverse discrete Fourier transform established in the previous subsection, every square integrable function on the interval $[-T, T]$ can be approximated arbitrarily well (in terms of L_2 convergence) by linear combinations of functions (4.10), which allows one to prove the general case of (4.9). ■

Theorem 4.5 shows that, for a given function $u \in L_2^m(\mathbb{R})$ the sequence of "incomplete" Fourier transforms U_T is a Cauchy sequence in $L_2^m(j\mathbb{R})$. Hence, there exists $U \in L_2^m(j\mathbb{R})$ such that $U - U_T| \to 0$ as $T \to \infty$ Any such function $U \in L_2^m(j\mathbb{R})$ is called a Fourier transform of u. Naturally, two different Fourier transforms of the same function $u \in L_2^m(\mathbb{R})$ must be equal almost everywhere on $j\mathbb{R}$.

The derivation of inverse Fourier transform on $L_2^m(j\mathbb{R})$ relies on minor modification of identity (4.8):

$$\frac{1}{2\pi}\int_{-\Omega}^{\Omega} |U(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} |u_\Omega(t)|^2 dt, \tag{4.11}$$

where

$$u_\Omega(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} e^{j\omega t} U(j\omega) d\omega.$$

Once again, the inverse Fourier transform $u \in L_2^m(\mathbb{R})$ of $U \in L_2^m(j\mathbb{R})$ is a limit of the Cauchy sequence u_Ω as $\Omega \to \infty$.

Theorem 4.6. *Function $U \in L_2^m(j\mathbb{R})$ is a Fourier transform of $u \in L_2^m(\mathbb{R})$ if and only if u is an inverse Fourier transform of U, in which case*

$$\int_{-\infty}^{\infty} |u(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(j\omega)|^2 d\omega. \tag{4.12}$$

4.2 Hardy Classes: H_2 and H_∞

Hardy classes H_2 are "one-sided" Fourier transforms of square integrable signals. They are introduced as subsets of $L_2^m(\mathbb{T})$ (in a "discrete time" framework) or $L_2^m(j\mathbb{R})$ (in a "continuous time" framework). However, the functions from H_2 are most interesting because of the possibility of "extending" them into the outside of the unit circle (for $L_2^m(\mathbb{T})$) or into the right half plane (for $L_2^m(j\mathbb{R})$), where they become analytical functions of complex argument.

The main objective of this section is to introduce functions from class H_∞, as a minor modification of class H_2, as they match exactly transfer functions of L2 gain stable causal

LTI systems, and also serve as building blocks for transfer functions of stabilizable causal LTI systems.

4.2.1 Analytical Functions

A function $F: \Omega \to \mathbb{C}$ defined on an open subset Ω of $\mathbb{C}$ is called *analytical* if the limit

$$\dot{F}(s) = \lim_{\delta \to 0} \frac{F(s+\delta) - F(s)}{\delta}$$

exists at all points $s \in \Omega$. As a rule, "elementary" functions, such as polynomials or exponents, or logarithms (within limited domains), or their compositions, turn out to be analytical. For example, function $F(s) = 1/s$ is analytical on $\Omega = \{s \in \mathbb{C}: \ s \neq 0\}$, and function

$$F(s) = \log(s) \stackrel{\text{def}}{=} \log|s| + \arg(s), \quad \arg(s) \in (-\pi, \pi)$$

is analytical in the right half plane

$$\mathbb{C}_+ = \{s: \ \mathrm{Re}\{s\} > 0\}.$$

In contrast, functions which require separating real and imaginary parts of s, such as $F(s) = \mathrm{Re}\{s\}$ or $F(s) = |s|$ are not analytical. For example, the expression

$$\frac{\mathrm{Re}\{s+\delta\} - \mathrm{Re}\{s\}}{\delta}$$

converges to 1 when δ approaches zero along the real axis, but converges to zero when δ approaches zero along the imaginary axis.

The following classical theorem gives an alternative description of analytical functions.

Theorem 4.7. *For some positive numbers R_0, R_1 such that $0 < R_0 < R_1$ let*

$$\Omega = \{s \in \mathbb{C}: \quad R_0 < |s| < R_1\}.$$

A function $F: \Omega \to \mathbb{C}^m$ is analytical if and only if there exists a function $f: \mathbb{Z} \to \mathbb{C}^m$ such that

$$\sum r^{-t}|f(t)| < \infty \quad \textit{for} \quad R_0 < r < R_1,$$

and

$$F(s) = \sum_{k=-\infty}^{\infty} s^{-t} f(t) \quad \forall \quad s \in \Omega.$$

4.2.2 Class $H_2^m(\mathbb{T})$

Formally speaking, $H_2^m(\mathbb{T})$ (or simply $H_2(\mathbb{T})$ when $m = 1$) is the subset of $L_2^m(\mathbb{T})$ consisting of those functions U for which the Fourier coefficients $u(t)$ equal zero for $t < 0$. In other words, $H_2^m(\mathbb{T})$ is the set of Fourier transforms of signals of finite energy.

Most significantly, for an arbitrary $U \in H_2^m(\mathbb{T})$ the infinite sum

$$U(w) = \sum_{t=\mathbb{Z}} u(t)w^{-t} = \sum_{t=0}^{\infty} u(t)w^{-t}$$

converges absolutely on the open set

$$\mathbb{D}^+ = \{w \in \mathbb{C} : \quad |w| > 1\}.$$

Allowing some abuse of notation, we will denote the resulting analytical function on $\mathbb{D}^+$ by the same letter, i.e. as $U : \mathbb{D}^+ \to \mathbb{C}$.

The following theorem shows that, combined with analiticity, boundedness of the energies $|U_r|^2$ for $U_r(z) = U(rz)$ over $r > 1$ yields a complete description of "extensions" of functions from class $H_2^m(\mathbb{T})$ to $\mathbb{D}^+$.

Theorem 4.8. *Let $U : \mathbb{D}^+ \to \mathbb{C}^m$ be a function. The following conditions are equivalent:*

(a) U is an extension of a function $U \in H_2^m(\mathbb{T})$;

(b) U is analytical on $\mathbb{D}^+$, and

$$\sup_{r>1} \int_{\mathbb{T}} |U(rz)|^2 dm(z) < \infty. \tag{4.13}$$

Moreover, when conditions (a),(b) are satisfied, the functions $U_r : \mathbb{T} \to \mathbb{C}^m$ defined for $r > 1$ by $U_r(z) = U(rz)$, converge to U, in the sense that $|U - U_r| \to 0$ as $r \to 1$.

One useful corollary of Theorem 4.8 is a criterion for a function U from class H_2^m to have zero Fourier coefficients $u(t) = 0$ for all $t < T$, where $T > 0$. Since this is equivalent to $V(s) = s^T U(s)$ being an extension of a function from class H_2, the necessary and sufficient condition is

$$\sup_{r>1} \int_{\mathbb{T}} r^{2T} |U(rz)|^2 dm(z) < \infty. \tag{4.14}$$

Example 4.1. *The function defined for* $|w| > 1$ *by* $U(w) = \log(1 - 1/w)$ *(where the values of the logarithm with imaginary part from* $(-\pi, \pi)$ *are used) is an extension of an* $H_2(\mathbb{T})$ *class function. To establish this, note that that* U *is analytical on* $\mathbb{D}^+$*, and use the inequality*

$$\left|1 - \frac{1}{z}\right| \leq 2\left|1 - \frac{1}{rz}\right| \quad \forall\ z \in \mathbb{T},\ r \geq 1.$$

Since the imaginary part of $\log(1 - 1/w)$ *for* $w \in \mathbb{D}^+$ *is bounded (lies within the* $(-\pi, \pi)$ *interval), and the real part cannot be larger than* $\log(2)$*, we have*

$$\left|\log(1 - \frac{1}{rz})\right| \leq \pi + \log(2) + \left|\log\left|1 - \frac{1}{z}\right|\right|,$$

which allows one to bound the integrals

$$\int_{\mathbb{T}} |\log(1 - 1/(rz))|^2\, dm(z) \leq \int_{\mathbb{T}} |\pi + \log(2) + |\log(1 - 1/z)||^2\, dm(z) < \infty$$

(the last inequality holds because $\log(|\Omega|)$ *is square integrable over* $\Omega \in (-\pi, \pi)$*).*

Naturally, the function $U(z) = \log(1 - 1/z)$ *(defined arbitrarily for* $z = 1$*) provides the values of* U *to the unit circle. Since we have already established an integrable majorant for* $|U(z) - U(rz)|$*, the convergence*

$$\lim_{r \to 1} \int_{\mathbb{T}} |U(z) - U_r(z)|^2 dm(z) = 0$$

follows from the point-wise convergence $U_r(z) \to U(z)$ *as* $r \to 1$ *for* $z \neq 1$*.*

Proof of Theorem 4.8, (a)⇒(b). Let $u \in L_2^m(\mathbb{Z})$ be the inverse Fourier transform of U. By assumption $u(t) = 0$ for $t < 0$. Hence the expansion

$$U(s) = \sum_{t \in \mathbb{Z}} u(t) s^{-t}$$

satisfies the absolute convergence condition

$$\sum_{t \in \mathbb{Z}} |u(t)| r^{-t} < \infty,$$

which makes U analytical on $\mathbb{D}^+$. Since $U_r(z) = U(rz)$ is the Fourier transform of $u_r(t) = r^{-t} u(t)$, the Parseval identity yields

$$\int_{\mathbb{T}} r^{2T} |U(rz)|^2 dm(z) = \sum_{t=0}^{\infty} r^{-2t} |u(t)|^2 \leq |U|^2.$$

Moreover, a similar use of the Parseval identity yields

$$\int_{\mathbb{T}} r^{2T}|U(rz)-U(z)|^2 dm(z) = \sum_{t=0}^{\infty}(1-r^{-t})^2|u(t)|^2 \to 0$$

as $r \to 1$. ■

Proof of Theorem 4.8, (b)⇒(a). By Theorem 4.7 there exists an expansion

$$U(s) = \sum_{t\in\mathbb{Z}} u(t)s^{-t}$$

which converges absolutely for $|s| > 1$. Hence the sequence $u_r(t) = r^{-t}u(t)$ is square summable for all $r > 1$, and, by the Parseval identity,

$$\int_{\mathbb{T}} r^{2T}|U(rz)|^2 dm(z) = \sum_{t=0}^{\infty} r^{-2t}|u(t)|^2$$

for all $r > 1$. Since the sum on the right converges to $+\infty$ as $r \to \infty$, unless $u(t) = 0$ for all $t < 0$, we conclude that $u(t) = 0$ for $t < 0$. Subject to this, the sum on the right converges to $|u|^2$ as $r \to 1$, which proves that $U \in H_2^m(\mathbb{T})$. ■

4.2.3 Class $H_\infty^{m,k}(\mathbb{T})$

Functions from class $H_\infty^{m,k}(\mathbb{T})$ will serve as transfer matrices of causal L2 gain stable LTI models (with m inputs and k outputs) in discrete time. Each element $G \in H_\infty^{m,k}(\mathbb{T})$ can be viewed as a k-by-m matrix with entries from $H_\infty(\mathbb{T}) = H_\infty^{1,1}(\mathbb{T})$. For the purpose of this section, it is sufficient to study the properties of the scalar functions $U \in H_\infty(\mathbb{T})$.

Formally, the class $H_\infty(\mathbb{T})$ is the subset of those functions $G \in H_2(\mathbb{T}) = H_2^1(\mathbb{T})$ which are uniformly bounded on $\mathbb{T}$, i.e. satisfy the condition

$$\sup_{z\in\mathbb{T}} |U(z)| < \infty.$$

The following statement describes the extensions of functions from $H_\infty(\mathbb{T})$ to $\mathbb{D}^+$.

Theorem 4.9. *Let $U : \mathbb{D}^+ \to \mathbb{C}$ be a function. The following conditions are equivalent:*

(a) U is an extension of a function $U \in H_\infty(\mathbb{T})$;

(b) U is analytical, and

$$\gamma = \sup_{z\in\mathbb{D}^+} |U(z)| < \infty. \tag{4.15}$$

Moreover, if conditions (a),(b) are satisfied then γ equals the essential supremum of $|U(z)|$ over $\mathbb{T}$, i.e. the minimal upper bound of those $h \geq 0$ for which the set of $z \in \mathbb{T}$ satisfying $|U(z)| \geq h$ has positive Lebesque measure.

Example 4.2. Out of the two functions

$$G_1(z) = \exp\left(\frac{1-z}{1+z}\right), \quad G_2(z) = \exp\left(\frac{z-1}{z+1}\right),$$

the first belongs to class $H_\infty(\mathbb{T})$ because it has an extension which is analytical and bounded in the domain $|z| > 1$. In contrast, function $G_2 \in L_2(\mathbb{T})$ does not belong to any of the classes $H_2(\mathbb{T})$ or $H_\infty(\mathbb{T})$, because it grows very large as z approaches -1 from the outside of the unit circle.

Proof of Theorem 4.9, (a)⇒(b). For every $s \in \mathbb{D}^+$ consider the continuous function $R_s : \mathbb{T} \to \mathbb{R}_+$ defined by

$$R_s(z) = 1 + \sum_{k=1}^{\infty} \left(\frac{z}{s} + \frac{\bar{z}}{\bar{s}}\right) = \frac{|s|^2 - 1}{|s-z|^2}.$$

By construction,

$$U(s) = \int_{\mathbb{T}} R_s(z) U(z) dm(z) \quad \forall \ \ s \in \mathbb{D}^+. \tag{4.16}$$

Since $R_s \geq 0$ and

$$\int_{\mathbb{T}} R_s(z) dm(z) = 1,$$

identity (4.17) implies

$$|U(s)| = \left| \int_{\mathbb{T}} R_s(z) U(z) dm(z) \right| \leq \int_{\mathbb{T}} R_s(z) |U(z)| dm(z) \leq \gamma$$

for all $s \in \mathbb{D}^+$. ■

Proof of Theorem 4.9, (a)⇒(b). For every $s \in \mathbb{D}^+$ consider the continuous function $R_s : \mathbb{T} \to \mathbb{R}_+$ defined by

$$R_s(z) = 1 + \sum_{k=1}^{\infty} \left(\frac{z}{s} + \frac{\bar{z}}{\bar{s}}\right) = \frac{|s|^2 - 1}{|s-z|^2}.$$

By construction,

$$U(s) = \int_{\mathbb{T}} R_s(z) U(z) dm(z) \quad \forall \ \ s \in \mathbb{D}^+. \tag{4.17}$$

Since $R_s \geq 0$ and

$$\int_{\mathbb{T}} R_s(z) dm(z) = 1,$$

identity (4.17) implies

$$|U(s)| = \left| \int_{\mathbb{T}} R_s(z) U(z) dm(z) \right| \leq \int_{\mathbb{T}} R_s(z) |U(z)| dm(z) \leq \gamma$$

for all $s \in \mathbb{D}^+$. ■

Proof of Theorem 4.9, (a)⇒(b). By Theorem 4.8, U is an extension of an $H_2(\mathbb{T})$ function V, and $|U_r - V| \to 0$ as $r \to 1$, where $U_r(z) = U(rz)$. It is sufficient to show that the essential supremum of V is not larger than γ. If this is not true, i.e. for some $\gamma_1 > \gamma$ the inequality $|V(z)| \geq \gamma_1$ holds on a subset $Z \subset \mathbb{T}$ of measure $\delta > 0$, then

$$|U_r - V|^2 = \int_{\mathbb{T}} |U_r(z) - V(z)|^2 dm(z) \geq \int_Z |U_r(z) - V(z)|^2 dm(z) \geq \delta(\gamma_1 - \gamma)^2 > 0,$$

which contradicts the convergence $|U_r - V| \to 0$ as $r \to 1$. ■

4.2.4 Class $H_2^m(j\mathbb{R})$

Formally speaking, $H_2^m(j\mathbb{R})$ (or simply $H_2(j\mathbb{R})$ when $m = 1$) is the subset of $L_2^m(j\mathbb{R})$ consisting of those functions U for which an inverse Fourier transform $u(t)$ equals zero for $t < 0$ (which is equivalent to *every* inverse Fourier transform u of U satisfying the condition $u(t) = 0$ for *almost all* $t < 0$). In other words, $H_2^m(j\mathbb{R})$ is the set of Fourier transforms of CT signals of finite energy.

Most significantly, for an arbitrary $U \in H_2^m(j\mathbb{R})$ the "Laplace transform" integral

$$U(s) = \int_{-\infty}^{\infty} u(t) e^{-st} dt = \sum_{t=0}^{\infty} u(t) e^{-st} dt$$

converges absolutely on the open set

$$\mathbb{C}^+ = \{s \in \mathbb{C} : \quad \mathrm{Re}\{s\} > 0\}.$$

Allowing some abuse of notation, we will denote the resulting analytical function on $\mathbb{C}^+$ by the same letter, i.e. as $U : \mathbb{C}^+ \to \mathbb{C}^m$.

The following theorem shows that, combined with analiticity, boundedness of the energies $|U_\sigma|^2$ for $U_\sigma(j\omega) = U(j\omega + \sigma)$ over $\sigma > 0$ yields a complete description of "extensions" of functions from class $H_2^m(j\mathbb{R})$ to $\mathbb{C}^+$.

Theorem 4.10. *Let $U : \mathbb{C}^+ \to \mathbb{C}^m$ be a function. The following conditions are equivalent:*

(a) U is an extension of a function $U \in H_2^m(j\mathbb{R})$;

(b) U is analytical, and

$$\sup_{\sigma>0} \int_{-\infty}^{\infty} |U(j\omega + \sigma)|^2 d\omega < \infty. \tag{4.18}$$

Moreover, when conditions (a),(b) are satisfied, the functions $U_\sigma : j\mathbb{R} \to \mathbb{C}^m$ defined for $\sigma > 0$ by $U_\sigma(j\omega) = U(\sigma + j\omega)$, converge to U, in the sense that $|U - U_\sigma| \to 0$ as $\sigma \to 0$.

One useful corollary of Theorem 4.10 is a criterion for a function U from class $H_2^m(j\mathbb{R})$ to be a Fourier transform of $u \in L_2^m(\mathbb{R})$ such that $u(t) = 0$ for all $t < T$, where $T > 0$. Since this is equivalent to $V(s) = \exp(Ts)U(s)$ being an extension of a function from class H_2, the necessary and sufficient condition is

$$\sup_{\sigma>0} e^{2\sigma T} \int_{-\infty}^{\infty} |U(\sigma + j\omega)|^2 d\omega < \infty. \tag{4.19}$$

Example 4.3. *The function defined for* $\mathrm{Re}\{s\} > 0$ *by* $U(s) = \log(s)/(s+1)$ *(where the values of the logarithm with imaginary part from* $(-\pi, \pi)$ *are used) is an extension of an* $H_2(j\mathbb{R})$ *class function.*

4.2.5 Class $H_\infty^{m,k}(j\mathbb{R})$

Functions from class $H_\infty^{m,k}(j\mathbb{R})$ will serve as transfer matrices of causal L2 gain stable LTI models (with m inputs and k outputs) in continuous time. Each element $G \in H_\infty^{m,k}(j\mathbb{R})$ can be viewed as a k-by-m matrix with entries from $H_\infty(j\mathbb{R}) = H_\infty^{1,1}(j\mathbb{R})$. For the purpose of this section, it is sufficient to study the properties of the scalar functions $U \in H_\infty(\mathbb{T})$.

Formally, the class $H_\infty(j\mathbb{R})$ can be defined as the subset of those functions $G : j\mathbb{R} \to \mathbb{C}$ which are uniformly bounded on $j\mathbb{R}$, and can be represented in the form

$$G(j\omega) = (1 + j\omega)\tilde{G}(j\omega) \tag{4.20}$$

for some $\tilde{G} \in H_2(j\mathbb{R})$.

Extending H_2 class function $\tilde{G}$ from (4.20) into the right half plane $\mathbb{C}^+$ defines a natural extension for $G \in H_\infty(j\mathbb{R})$, via $G(s) = (1 + s)\tilde{G}(s)$. The following statement describes the extensions of functions from $H_\infty(j\mathbb{R})$ to $\mathbb{C}^+$.

Theorem 4.11. *Let $U : \mathbb{C}^+ \to \mathbb{C}$ be a function. The following conditions are equivalent:*

(a) U is an extension of a function $U \in H_\infty(j\mathbb{R})$;

(b) U is analytical, and

$$\gamma = \sup_{s \in \mathbb{C}^+} |U(s)| < \infty. \tag{4.21}$$

Moreover, if conditions (a),(b) are satisfied then γ equals the essential supremum of $|U(s)|$ over $s \in j\mathbb{R}$, i.e. the minimal upper bound of those $h \geq 0$ for which the set of $\omega \in \mathbb{R}$ satisfying $|U(j\omega)| \geq h$ has positive Lebesque measure.

4.3 L2 Gain Stable Transfer Matrix Models

Defining LTI system models in terms of their transfer functions is supposed to be straightforward: apply Fourier transform to the input, multiply the result by the transfer function, and then apply inverse Fourier transform to the product. There are several important questions to resolve, though:

(a) What are the boundary conditions for the resulting system?

(b) What to do when the signals involved are not square integrable

(c) Which conditions guarantee that the resulting system is well defined (e.g. the inverse Fourier transform involved does exist)?

In this section we use functions from Hardy classes H_∞ as transfer matrices L2 gain stable transfer matrix models.

The following shortcut notation (applicable to signals of arbitrary dimension d), where $X = \mathbb{Z}$, $Y = \mathbb{T}$, and $Z = \ell$ for discrete time systems, or $X = \mathbb{R}$, $Y = j\mathbb{R}$, and $Z = \mathcal{L}$ for CT systems) will be used in this section:

- $\mathcal{F}$: Fourier transform $\mathcal{F}: \; L_2^d(X) \to L_2^d(Y)$;
- $\mathcal{F}^{-1}$: inverse Fourier transform $\mathcal{F}^{-1}: \; L_2^d(Y) \to L_2^d(X)$;
- P_T: "past projection" $P_T: \; Z^d \cup L_2^d(X) \to L_2^d(X)$,

$$P_T f = g \quad \text{means} \quad g(t) = \begin{cases} f(t), & 0 \leq t \leq T, \\ 0, & \text{otherwise;} \end{cases}$$

- P_+: "future projection" $P_+: \; L_2^d(X) \to L_2^d(X)$,

$$P_+ f = g \quad \text{means} \quad g(t) = \begin{cases} f(t), & t \geq 0, \\ 0, & t < 0; \end{cases}$$

4.3.1 Real Symmetry

So far, we have applied Fourier transforms to general complex-valued functions and sequences. Since we use real-valued functions to describe actual "real world" signals, a description of Fourier transforms of real-valued functions will be useful.

If $U \in L_2^m(\mathbb{T})$ is a Fourier transform of $u \in L_2^m(\mathbb{Z})$ then the function $U_c \in L_2^m(\mathbb{T})$ defined by $U_c(z) = \overline{U(\bar{z})}$ is a Fourier transform of $u_c \in L_2^m(\mathbb{Z})$ defined by $u_c(t) = \overline{u(t)}$. Since $U_c(z) = U(z)$ for almost all $z \in \mathbb{T}$ if and only if $u_c(t) = u(t)$ for all $t \in \mathbb{Z}$, inverse Fourier transform of a function $U \in L_2^m(\mathbb{T})$ is real-valued if and only if $U(\bar{z}) = \overline{U(z)}$ for almost all $z \in \mathbb{T}$. Similarly, inverse Fourier transform of a function $U \in L_2^m(j\mathbb{R})$ is real-valued if and only if $U(-j\omega) = \overline{U(j\omega)}$ for almost all $\omega \in \mathbb{R}$.

This observation suggests that transfer matrices of LTI systems should also have the real symmetry property, which leads us to the definition of class $H_{\infty|Re}^{k,m}(X)$ (where $X = \mathbb{T}$ or $X = j\mathbb{R}$) as the subset of those functions G from $H_\infty^{k,m}(X)$ (i.e. k-by-m matrices with entries from the class $H_\infty(X)$) which satisfy the *real symmetry* condition $G(\bar{z}) = \overline{G(z)}$. Recall that a rational function satisfies the real symmetry condition if and only if it has real coefficients. Naturally, one expects transfer matrices of "physically implementable" systems to have real symmetry.

4.3.2 Discrete Time Stable Transfer Matrix Models

We will use the following preliminary observation.

Theorem 4.12. *For every $G \in H_{\infty|Re}^{k,m}(\mathbb{T})$ and $w \in \ell^m$ there exists a unique $y \stackrel{\text{def}}{=} L_G w \in \ell^k$ such that*

$$P_T y = P_T \mathcal{F}^{-1} G \mathcal{F} P_T w \quad \forall\ T \geq 0. \tag{4.22}$$

Moreover, if $w(t) = 0$ for $t \leq \tau$ then $y(t) = 0$ for $t \leq \tau$.

The function L_G defined in Theorem 4.12 generates the "zero-state" response of the transfer matrix model defined by G. Its definition follows the familiar "Fourier tranform, multiplication, inverse Fourier transform" pattern, with projections P_T inserted to ensure legality of the associated operations.

Also, for $G \in H_{\infty|Re}^{k,m}(\mathbb{T})$, let X_G denote the subset of sequences $x_0 \in L_2^k(\mathbb{Z})$ which can be approximated arbitrarily well (in the sense of $|x_0 - y_0| \to 0$ by sequences y_0 of the form $y_0 = P_+ \mathcal{F}^{-1} G \mathcal{F} w_0$ (i.e. those obtained by taking $w_0 \in L_2^m(\mathbb{Z})$, applying Fourier transform, multiplying the result by G, applying inverse Fourier transform, and then zeroing out the samples with $t < 0$), where $w_0 \in L_2^m(\mathbb{Z})$ is such that $w_0(t) = 0$ for $t \geq 0$, and $w_0(t) \in \mathbb{R}^m$ for $t < 0$. Due to the real symmetry assumption, all sequences in X_G

are real. The set X_G will be used to represent the "boundary conditions" or "zero input response" of the LTI system associated with G.

Finally, the *transfer matrix model* $S_{DT}^{TM}(G)$ defined by $G \in H_{\infty|Re}^{k,m}(\mathbb{T})$ is the discrete time system $S_{DT}^{TM}(G) \in \mathcal{S}_{DT}^{m,k}(X_G)$, such that

$$S_{DT}^{TM}(G)(w, x_0) = \{x_0 + L_G w\} \quad \forall\ w \in \ell^m,\ x_0 \in X_G. \tag{4.23}$$

Example 4.4. Let $G:\ \mathbb{T} \to \mathbb{C}$ be defined by $G(z) = 1/z$. Since G is the Fourier transform of real-valued signal

$$g(t) = \begin{cases} 1, & t = 1, \\ 0, & t \neq 1, \end{cases}$$

we conclude that $G \in H_{\infty|Re}(\mathbb{T})$. What kind of LTI system is $S_{DT}^{TM}(G)$?

Let us begin by studying time-domain implications of the Fourier transform relation $Y(z) = G(z)W(z)$, where $Y, W \in L_2(\mathbb{T})$. Since $Y(z) = G(z)W(z)$ means $zY(z) = W(z)$, the corresponding inverse Fourier transforms y, w satisfy the difference equation $y(t+1) = w(t)$ for all $t \in \mathbb{Z}$.

The original observation proves that $y = L_G w$ means

$$y(t) = \begin{cases} w(t-1), & t > 0, \\ 0, & t = 0. \end{cases}$$

Similarly, sequences of the form $y_0 = P_+ \mathcal{F}^{-1} G \mathcal{F} w_0$ are such that

$$y_0(t) = \begin{cases} w_0(t-1), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

where $w_0(t) = 0$ for $t \geq 0$, which means that $y_0(t) = 0$ for all t except at $t = 0$, while $y_0(0)$ can be arbitrary. Therefore the set X_G consists of all sequences $x_0:\ \mathbb{Z} \to \mathbb{R}$ such that $x_0(t) = 0$ for all $t \neq 0$. While, formally speaking, X_G is a set of sequences, its elements can be parametrized linearly by a single real parameter!

Finally, combining these descriptions of L_G and X_G, we conclude that the set of input-output pairs (w, y) in system $S_{DT}^{TM}(G)$ is defined by equation $y(t+1) = w(t)$, where the initial value $y(0)$ can be an arbitrary real number, and is determined by the boundary condition vector $x_0 \in X_G$. No surprises here!

Proof of Theorem 4.12 To simplify notation and terminology, the proof is given for the case $m = k = 1$.

For every $u \in \ell$ and $T > 0$ the Fourier transform U_T of $u_T = P_T u$ is real symmetric, and has an analytical extension $\tilde{U}_T:\ \mathbb{D}^+ \to \mathbb{C}$ such that

$$\lim_{r \to 1} \sup_{z \in \mathbb{T}} |\tilde{U}_T(rz) - U_T(z)| = 0.$$

As an element of $H_\infty(\mathbb{T})$, function G also has an analytical extension $\tilde{G}: \mathbb{D}^+ \to \mathbb{C}$ which is uniformly bounded, and such that

$$\lim_{r\to 1}\int_{\mathbb{T}} |\tilde{G}(rz) - G(z)|^2 dm(z) = 0.$$

Hence the function $\tilde{Y}_T: \mathbb{D}^+ \to \mathbb{C}$ defined by $\tilde{Y}_T(z) = \tilde{G}(z)\tilde{U}_T(z)$ is an analytical extension of $Y_T = GU_T$, and

$$\begin{aligned}
\int_{\mathbb{T}} |Y_T(z) - \tilde{Y}_T(rz)|^2 dm(z) &= \int_{\mathbb{T}} |G(z)U_T(z) - \tilde{G}(rz)\tilde{U}_T(rz)|^2 dm(z) \\
&= 2\int_{\mathbb{T}} \{|\tilde{G}(z)(U_T(z) - \tilde{U}_T(rz))|^2 + |(G(z) - \tilde{G}(rz))\tilde{U}_T(rz)|^2\} dm(z) \\
&\le 2\max_{z\in\mathbb{T}} |U_T(z) - \tilde{U}_T(rz)|^2 \int_{\mathbb{T}} |G(z)|^2 dm(z) \\
&\quad +2\max_{z\in\mathbb{T}} |G(z)|^2 \int_{\mathbb{T}} |G(z) - \tilde{G}(rz)|^2 dm(z) \to 0
\end{aligned}$$

as $r \to 1$. Hence $Y_T \in H_2(\mathbb{T})$ has inverse Fourier transform y_T. To complete the proof, it is sufficient to show that for every $\tau > T$ the equality $y_T(t) = y_\tau(t)$ holds for all $t \le T$. Indeed, since $u_T(t) = u_\tau(t)$ for $t \le T$ we have

$$\sup_{r>1} r^{2T} \int_{\mathbb{T}} |\tilde{U}_T(rz) - \tilde{U}_\tau(rz)|^2 dm(z) < \infty.$$

Since $\tilde{G}(z)$ is uniformly bounded in $\mathbb{D}^+$, this implies

$$\sup_{r>1} r^{2T} \int_{\mathbb{T}} |\tilde{Y}_T(rz) - \tilde{Y}_\tau(rz)|^2 dm(z) = \sup_{r>1} r^{2T} \int_{\mathbb{T}} |\tilde{G}(rz)(\tilde{U}_T(rz) - \tilde{U}_\tau(rz))|^2 dm(z) < \infty,$$

and hence $y_T(t) = y_\tau(t)$ for $t \le T$. ■

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4.3.3 Continuous Time Stable Transfer Matrix Models

The development follows the DT case closely. We start by defining zero state response.

Theorem 4.13. *For every $G \in H^{k,m}_{\infty|Re}(j\mathbb{R})$ and $w \in \mathcal{L}^m$ there exists a unique $y \stackrel{\text{def}}{=} L_G w \in \mathcal{L}^k$ such that*

$$P_T y = P_T \mathcal{F}^{-1} G \mathcal{F} P_T w \quad \forall\ T \ge 0. \tag{4.24}$$

Moreover, if $w(t) = 0$ for $t \le \tau$ then $y(t) = 0$ for $t \le \tau$.

For $G \in H^{k,m}_{\infty|Re}(\jmath\mathbb{R})$, let X_G denote the subset of sequences $x_0 \in L^k_2(\mathbb{R})$ which can be approximated arbitrarily well (in the sense of $|x_0 - y_0| \to 0$ by sequences y_0 of the form $y_0 = P_+\mathcal{F}^{-1}G\mathcal{F}w_0$ (i.e. those obtained by taking $w_0 \in L^m_2(\mathbb{R})$, applying Fourier transform, multiplying the result by G, applying inverse Fourier transform, and then zeroing out the samples with $t < 0$), where $w_0 \in L^m_2(\mathbb{Z})$ is such that $w_0(t) = 0$ for $t \geq 0$, and $w_0(t) \in \mathbb{R}^m$ for $t < 0$. Due to the real symmetry assumption, all sequences in X_G are real.

Finally, the *transfer matrix model* $S^{TM}_{CT}(G)$ defined by $G \in H^{k,m}_{\infty|Re}(\jmath\mathbb{R})$ is the continuous time system $S^{TM}_{CT}(G) \in \mathcal{S}^{m,k}_{CT}(X_G)$, such that

$$S^{TM}_{CT}(G)(w, x_0) = \{x_0 + L_G w\} \quad \forall\ w \in \ell^m,\ x_0 \in X_G. \tag{4.25}$$

Example 4.5. Let $G:\ j\mathbb{R} \to \mathbb{C}$ be defined by $G(s) = e^{-\tau s}$, where $\tau > 0$ is a parameter. Since G is uniformly bounded on $j\mathbb{R}$, and $G(s)/(s+1)$, as the Fourier transform of

$$g(t) = \begin{cases} 0, & t < \tau, \\ e^{\tau - t}, & t \geq \tau, \end{cases}$$

is from class $H_2(j\mathbb{R})$, we conclude that $G \in H_\infty(j\mathbb{R})$. Moreover, since G has real symmetry, $G \in H_{\infty|Re}(j\mathbb{R})$. What kind of LTI system is $S^{TM}_{CT}(G)$?

We begin by studying time-domain implications of the Fourier transform relation $Y(j\omega) = G(j\omega)W(j\omega)$, where $Y, W \in L_2(j\mathbb{R})$: $y(t) = w(t - \tau)$ for all $t \in \mathbb{R}$.

The original observation proves that $y = L_G w$ means

$$y(t) = \begin{cases} w(t - \tau), & t \geq \tau, \\ 0, & t < \tau. \end{cases}$$

Similarly, functions of the form $y_0 = P_+\mathcal{F}^{-1}G\mathcal{F}w_0$ are such that

$$y_0(t) = \begin{cases} w_0(t - \tau), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

where $w_0(t) = 0$ for $t \geq 0$, which means that $y_0(t) = 0$ for all $t \geq 0$, and the restriction of y_0 to the interval $(0, \tau)$ can be arbitrary square integrable function. Therefore the set X_G consists of all square integrable functions $x_0:\ \mathbb{R} \to \mathbb{R}$ such that $x_0(t) = 0$ for all $t \geq \tau$. Note that the set of "boundary conditios" does not have a finite basis: this is what we expect when the transfer function is not rational.

Finally, combining these descriptions of L_G and X_G, we conclude that the set of input-output pairs (w, y) in system $S^{TM}_{CT}(G)$ is defined by equation $y(t + \tau) = w(t)$, where the value $y(t)$ for $t < \tau$ can be an arbitrary, as long as they are real, and the resulting function is square integrable over $(0, \tau)$.